TMM021: Mathematics in Elementary Education I (Illustrations)

Typical Range: 4-5 Semester Hours Recommendation:

It is essential for all teachers of mathematics to understand the reasoning underlying the mathematics they are teaching. They need to understand why various procedures work, how each idea they will be teaching connects with other important ideas in mathematics, and how these ideas develop and become more sophisticated. Furthermore, knowing only the mathematics of the elementary grades is not sufficient to be an effective teacher of elementary grades mathematics. Neither is it sufficient to require that future teachers simply "take more math courses." This document describes the kinds of mathematics and mathematical experiences that we believe are essential for their mathematical learning and professional development. Exploration in middle grade topics are introduced at the elementary level to begin planting the seed in preparation for advanced teachings.

We take the view that mathematics courses for future teachers must prepare them to do a different kind of work in their mathematics teaching than they likely experienced in their own schooling. With this in mind, we encourage oral and written communication in these classes as both a learning tool and as preparation for handling mathematical questions which arise in their classrooms. For example, we aim for discussion that focuses on the deep mathematical reasoning underlying the computational procedures that are usually taught in elementary school. We recommend this be done by exploring common misconceptions with preservice teachers and by engaging future teachers in activities that require them to interpret their own multiple ways of addressing questions and interpreting children's work which might be incorrect, incomplete, or different from their adult ways of thinking. We aim to encourage a serious approach to the mathematics through questions (by instructors *and* preservice teachers) about *why* we do things the way we do, what the operations *mean*, what the *units* are on the answers, and how the mathematical ideas of the day *connect* to other mathematical ideas (looking for the "big ideas" in a problem).

We recommend these courses be activity based so that opportunities for deep, connected learning arise while misconceptions are addressed. This requires "good" problems and "good" questioning by instructors. A good problem needs to engage the preservice teachers at an appropriate level of challenge (hard enough that the preservice teacher cannot answer on autopilot). Often this is accomplished by confronting them with misconceptions framed as "a child said this…" and directions to analyze and/or justify the result. The justifications can themselves become a source of deep discussion - one preservice teacher may not understand another's solution or explanation, a preservice teacher's correct answer may have a flawed explanation, or a new method may be generated once an array of other methods has been shared. Sometimes the discussion is generated when an instructor says, "I want to list all of the different answers we got before we discuss the reasoning" (which means the class can actually discuss their own wrong answers). All of the above takes a lot of time, so we recommend 8 to 10 credit hours of such work in teacher education programs or prerequisites.

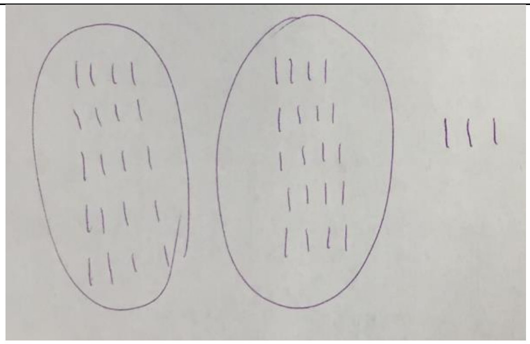
At the same time, we strongly urge that instructors be aware that these are college level mathematics courses. “A credit-bearing, college-level course in Mathematics must use the standards required for high school graduation by the State of Ohio as a basis and must do at least one of the following: 1) broaden, or 2) deepen, or 3) extend the student’s learning.” We recommend dedicated coursework for this content because there is much deep mathematics to be explored in understanding what we loosely term “elementary school mathematics.” This is a mathematical content course. The learning outcomes are all focused on using, justifying, and connecting mathematical concepts and do not address “how to teach.” It is sometimes appropriate to discuss topics which are more directly relevant to a methods course when they serve the purpose of motivating a mathematical discussion, but students should not be assessed on methods in these courses.

The courses should integrate reasoning, flexibility, multiple explanations, and number sense. Leading questions help students to make connections among topics and to develop their own questioning skills (e.g. Have we seen this idea before? How are these two different solution methods related to each other? What does it mean? How do you know? Could you draw a picture to show it? Where did they go wrong? Could we use their idea in this other problem?) Students should understand that mathematics is correct if it makes personal sense and if one can explain it in a way to make sense to others, not if an authority certifies it. Preservice teachers should leave these courses knowing that math makes sense and armed with the underlying knowledge they need to make math make sense for their future students. Elementary students will often come up with their own creative approaches to problems, so future teachers must be able to evaluate their mathematical viability *before* deciding how to respond instructionally. Squashing a child’s idea can be quite harmful. And often children’s ideas are right or almost right.

The learning outcomes that follow are all viewed as essential by the committee (marked with an asterisk) and demonstrate the level of student engagement motivating this course. In addition, learning outcomes are not specific items/topics but rather learning outcomes of course entirety. Institutional courses should provide an integrated experience with learning outcomes woven together throughout the course.

Learning Outcomes		**Illustrations**
1. Numbers	**1a.** Discuss the intricacies of learning to count, including the distinction between counting as a list of numbers in order and counting to determine a number of objects, and use pairings between	● Write a mock classroom interaction showing how a young child might miscount a set of objects by double counting, skipping objects, etc. ● Write commentary on fake student work dealing with this topic (5 objects are shown, the student has marked 1, 2, __, 3, 4, and claims there are 4 objects). ● Argue based on pairing that two sets have the same cardinality without pairing with the list of number words. For instance, explain whether you can tell if there are the same number of people in the classroom as there are cookies without enumerating both sets. For instance, if each student has 1 and only 1 cookie, then there are an equal number of students and cookies. However, if each student has at least 1, but

	elements of two sets to establish equality or inequalities of cardinalities. *	some have more than 1, then there are more cookies. If each student has 1 or none, then there are less cookies. ● Discuss struggles children may have with "teen" numbers. For example, 17 is read as seventeen while 27 is read as twenty-seven. This is difficult for children who are just beginning to read and learning to always read left to right.
	1b. Attend closely to units (e.g., apples, cups, inches, etc.) while solving problems and explaining solutions. *	● Using base 10 blocks (or objects) choose one of the pieces as a whole and have students identify the relative value of each of the other pieces. Make sure to pick pieces that will identify both larger and decimal size pieces. This idea is useful to address learning outcome 2g. ● See the website below for a variety of images that can often be counted in more than one way. https://mathforlove.com/lesson/unit-chats/ ● "A serving of oatmeal is ¾ of a cup of oats. How many servings is 2 cups of oats?" - questions like this which involve switching between two related units (servings vs. cups) are critical.
	1c. Discuss how the base-ten place value system (including extending to decimals) relies on repeated bundling in groups of ten and how to use objects, drawings, layered place value cards, base-ten blocks, and numerical expressions (including integer exponents) to help reveal base-ten structure. *	● Organize toothpicks and baggies, straws and boxes, as well as base ten blocks. For example, have a group of students count a large pile of loose toothpicks (e.g., 452) and then arrange them on the table in a way that will allow their classmates to see how many toothpicks are there in one glance. Answer: This should promote a canonical base ten arrangement of 10 single toothpicks into bundles of 10 by either putting a rubber band around each set of ten or by enclosing 10 loose toothpicks in a baggie. One hundred could be shown by putting 10 baggies or bundles into a larger baggie or (super-bundle). ● Have students reorganize a collection of objects which are not bundled in a way that conforms with base 10. Example pictures: 

		 ● Start with a collection which is bundled in a base 10 conforming manner and unbundle it in various non-conforming manners (this supports mental math strategies later) ○ Example answer for the number 459: this can be expressed as 450 + 9 or 250 + 200 + 9 or 225 + 225 + 9, etc. ● Compare sets of bundles and make inequality statements. ● Explore children's errors. ○ Thirty-eight, thirty-nine, thirty-ten, thirty-eleven, ○ 19 > 23 since ten pieces is more than 5 pieces. ● See the website below for online layered place value cards and base ten blocks (and other great manipulatives). http://toytheater.com/category/teacher-tools/ ● See the website below for base ten blocks (number pieces) on an app where you can create an assignment, share it with students, and after students are finished, they can share the results with you: https://www.mathlearningcenter.org/resources/apps/number-pieces ● See the website below for ten frames (number frames) on an app where you can create an assignment, share it with students, and after students are finished, they can share the results with you: https://www.mathlearningcenter.org/resources/apps/number-frames

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1d. Use the CCSS (Common Core State Standards) development of fractions: *

- Start with a whole.
- Understand the fraction 1/b as one piece when the whole is divided into b equal pieces.
- Understand the fraction a/b as a pieces of size 1/b and that the fraction a/b may be larger than one.
- Understand fractions as numbers that can be represented in a variety of ways, such as with lengths (esp. number lines), areas (esp. rectangles), and sets (such as a collection of marbles).
- Use the meaning of fractions to explain when two fractions

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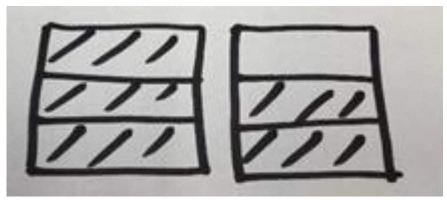
- Write an expression that shows how you would solve the following problem: *Ms. Smith decided that 35 apples would be enough to feed one third of the class. How many apples will she need to buy to feed her whole class?*
  - Possible answers: 35 + 35 + 35 This is an addition problem because she will need to combine 35 apples for the first third of the class with another 35 apples for the second third of the class and then 35 apples for the last third. OR 3 x 35 Because she will need three groups of 35, this is a multiplication problem. OR this problem could be solved using a proportion such as 35/x = ⅓ OR 35 = (⅓)X but this would only be appropriate for upper elementary students.
- Explain how the following figure could illustrate 5/3 or ⅚ and that the context of the situation would determine which answer is correct.



- Use drawings and reasoning to solve problems and explain solutions. For example: “One serving of rice is ½ cup. You ate ⅔ of a cup of rice. How many servings did you eat?”
  - Examine and critique reasoning, such as “A student said that 2/3 of a cup of rice is 1 serving plus another ⅙. Is that correct?”
  - It is 1 serving plus another ⅙ of a cup of rice, but the ⅙ of a cup of rice is ⅓ of a serving. That is because 1/2 = 3/6. The ⅙ of a cup of rice is one of the 3/6 of a cup that make a ½ cup serving.
- We have a cake. ⅓ of the cake is vanilla. Joe eats 2/5 of the vanilla part of the cake. What fraction of the cake did Joe eat?

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	are equivalent.	○ Note that this does involve fraction multiplication, but it can be solved directly using the meaning of fractions without reference to multiplication. Attending carefully to the different wholes involved in the problem (whole cake vs. vanilla part). ● Locate 7/5 on the number line. Conversely, here are the locations of 0 and 7/5 on the number line: find the location of 1. ○ Locate 7/5 on the number line: This might be done by recognizing that 7/5 means “7 pieces of size ⅕”, so there should be 7 equal intervals between 0 and 7/5 on the number ○ Here are the locations of 0 and 7/5 on the number line: find the location of 1. This might be done by recognizing that 7/5 means “7 pieces of size ⅕”, so there should be 7 equal intervals between 0 and 7/5 on the number line. 5 of these intervals would show a length of 1. ● Give examples showing that $\frac{a}{b}$ and “a out of b” can mean different things. ○ My sister just came home from a party. She said she ate ⅓ of the pizza at the party. Does this mean that she ate 1 whole pizza and there were 3 pizzas available at the party? Answer: No, there might have been only one pizza (and she ate ⅓ of that pizza, sharing it equally with two friends). It also could mean that she ate 1 of three equal-sized pizzas that were available, or that she ate 4 of 12 equal pizzas that were available (and so forth). But what if the pizzas were different sizes? If she ate the largest (or smallest or …) pizza, then she did not eat ⅓ of the total amount of pizza. ● For equivalent fractions, students can use paper folding to reason about why the “numerator and denominator are multiplied by the same number.” ○ For example, have students fold and color a piece of paper to show ⅔ of the paper. Then have the students fold this SAME piece of paper into twelfths. What happened to each of the original thirds? How many sections of size 1 twelfth are in each third? How many twelfths make up the same fraction (⅔) of the original paper? Could you have predicted this number?

		○ Repeat the activity imagining that you start with a paper that is folded into twenty-sixths. If we started with 3/26 shaded, how many 1/104ths would be shaded in the same paper? How do you know? Answer: this discussion can focus on the fact that a paper originally folded into 3 equal parts can now be folded into 12 equal parts by folding each original third into 4 equal parts (so we have 3x4 equal parts in the same whole original paper), then we can be certain that each of the original shaded parts will now be in four equal parts as well, so each of the two shaded thirds is now made up of four shaded pieces of size 1/12, so the numerator of the new picture is 2 x 4. Moving to any larger number helps us see the pattern of our actions that is expressed in our rule that ⅔ = (2 x 4)/ (3 x 4).
	1e. Model positive versus negative numbers on the number line and in real-world contexts. *	● Create a "trip line" (see Bob Moses curriculum) Students define a trip. The initial location is 0. Each stop is ordered and moving in the opposite direction is noted as negative. The number line acts as a map of locations. ● Walking the number line. How can you start at 7 on the number line and get to -2 on the number line? ● Debit and credit/loans examples could be used. ● Thermometers provide a real-world example of negative values. ● In some games (e.g. Jeopardy) negative scores can occur.
	1f. Reason about the comparison (=, <, >) of numbers across different representations (such as fractions, decimals, mixed numbers, ...). *	● Example activity: sets of number cards can be given to groups of students who order the cards from least to greatest (or greatest to least) and then explain their orderings to other groups. Example of one set of cards: -⅖, 3/5, 1.2, 4/3, -0.3, 10% of 14, 14% of 10. ● The fraction/decimal/percent tower is a good physical manipulative for this. ● The website www.toytheater.com has several different representations of numbers that can be compared online.
	1g. Demonstrate the skill of calculating simple	● One can multiply by 5 by multiplying by 10 and dividing by 2.

	arithmetic problems WITHOUT the use of a calculator. *	● Students should be able to calculate “easy” percents mentally. E.g. The check is $25.80. How much is a 20% tip? ● Students should be able to compute “easy” percentages involving percent increase or decrease without the use of a calculator. For example: If a coat normally costs $40 but there is a 35% off sale, how much will the coat cost? ● Students should be able to solve the problem for themselves and also to analyze some given solutions to determine whether the given solution methods were valid. For example: ○ Student A may argue using mental math that since 10% of 40 is 4, then 20% is 8, and 30% is 12 and 5% is 2, so 35% of 40 is 14. Since the coat is 35% off, the cost is $26. ○ Student B might use pen and paper and compute 0.65 40 = 26, so the cost is $26. ○ Student C may do the computation 4 6.5 by doubling 6.5 to get 13 and doubling again to get $26. ○ Student D may say that since 25% of 40 is 10 and 10% of 40 is 4, then $14 is subtracted to get $26. Any student method should be accepted as long as the student can explain his/her method to others and use it appropriately in similar situations with different numbers. ● Students should be able to explain general results such as: If the price of a radio is increased by 10% and the new price is decreased by 10%, is the final price lower, the same as, or higher than the original price? ○ When presented this question in a multiple-choice format as above, many students will initially select that the final price is the same as the original. If prompted that this is incorrect, they may select an example price, do the calculations, and realize that in that instance, the final price is lower than the original price. Some students will pick another example price to see whether it is again lower. But eventually students should be able to explain WHY it is lower, not just give examples that support this as sometimes selecting examples won’t give the full picture. ● Students should be presented with problems that are given in a variety of formats. Such as:

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Multiple Choice: Select the largest fraction below if N is a natural number that is greater than one.
(a) 1/N (b) 1/(N-1) (c) 1/(N+1) (d) 1/N^2
Ordering: Order the fractions below from smallest to largest if N is a natural number that is greater than one.
(a) 1/N (b) 1/(N-1) (c) 1/(N+1) (d) 1/N^2
Always, sometimes, never: If a and b are natural numbers,
then a/b x 5,471 < 5,471 (select the best answer choice)
Always Sometimes Never
When True/False questions are used, students can be challenged to justify results that they believe are true and to create counterexamples for results that they believe are false. They may be further challenged to fix a false statement to make it true.
In some cases, an Always, Sometimes, Never format or a select all the correct answers format may be displayed in a table.

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<tr><td>n is a natural number</td><td>Less than 3,472</td><td>Equal to 3,472</td><td>Greater than 3,472</td></tr>
<tr><td>The product of 3,472 and n can be</td><td></td><td></td><td></td></tr>
<tr><td>The product of 3,472 and 1/n can be</td><td></td><td></td><td></td></tr>
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<td>Operations</td>
<td>2a. Recognize addition, subtraction, multiplication, and division as descriptions of certain types of reasoning and correctly</td>
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I had 12 jellybeans. My brother gave me some more, and now I have 18 jellybeans. How many jellybeans did I get from my brother?

This could be solved by adding on or by taking away. For instance, you count on : “13 (1 finger ), 14 (2 fingers ), 15 (3 fingers ), 16 (4 fingers

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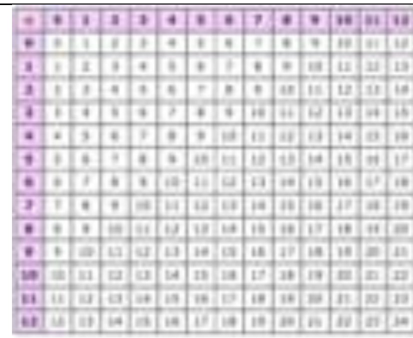
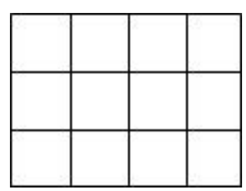
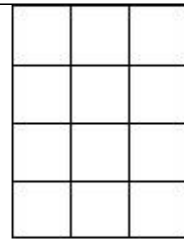
	use the language and notation of these operations. *	), 17 (5 fingers ), 18 (6 fingers  ) ” or taking away “18 minus 12 is 6”. Note – students may incorrectly use the counting on strategy here if they begin the count with the original number and the 1 finger picture. ● Subtraction is NOT always “take-away.” Subtraction can also be used in comparison, partitioning, or even additive situations. For example, *I need $64 to buy a bike. I already have $48 saved. How much more do I need?* This can be illustrated with the missing addend equation 48 + __ = 64, but the problem may be solved with subtraction. ● See Page 94 of the 2017 Ohio’s Learning Standards: Mathematics for common addition and subtraction situations. ● See Page 95 of the 2017 Ohio’s Learning Standards: Mathematics for common multiplication and division situations. ● Unknowns can occur in all positions of an equation.
	2b. Illustrate how different problems are solved by addition, subtraction, multiplication and division and be able to explain how the operation used is connected to the solving of the problem. *	● Students should be able to identify and create problems which call for use of each of the 4 operations, or combinations of operations, and use all of the different representations of numbers. Here is an example problem involving addition, multiplication, fractions, and decimals: ○ “I had $203133.34 in my bank account. Over the course of the year, I was able to put another $23,145.21 into the account. I then spent 2/5 of my money on a house. How much money did I spend on the house?” ● Here is an example which involves division of fractions (a notoriously difficult concept): ○ “A jug holds 11/6 of a gallon of milk. What fraction of a jug is filled if you pour 3/2 of a gallon into it?”
	2c. Recognize that addition, subtraction, multiplication, and	● This is partially addressed in the examples above. Working with real world problems which can be modeled using these operations is important.

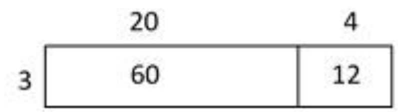
	division problem types and associated meanings for the operations (e.g., CCSS, pp. 88–89) extend from whole numbers to fractions and decimals. *	● It is important to carefully develop these ideas and make the connections. For instance, drawing area models of 2 x 3 and 2 x 4 and then asking the students to draw area models of 2 x 3.1, 2 x 3.2, …, 2 x 3.9 and make sense of these. Recognize that the picture for 2 x 3.2 is also a picture for 2 x 3⅕. ● If students are confused about which operation to use in a word problem, it sometimes helps to replace the numbers with whole numbers which are easier to calculate with, and then reason by analogy. For instance, in the "jug problem" in the illustrations for 2b, you could say "What if a jug holds 5 gallons of milk, and you poured 3 gallons in? What operation would you use to find the fraction of a jug which is filled? We should use the same operation in the original problem. "
	2d. Employ teaching/learning paths for single-digit addition and associated subtraction and single-digit multiplication and associated division, including the use of properties of operations (i.e., the field axioms). *	● Math Talks or number talks are a great way to get students to see a variety of ways to think about problems. This would include decomposition and composition. For example, you could ask students to calculate 7*8, and ask students to explain their reasoning. You might find one student who knows 7*5 = 35 and 7*3 = 21, so 7*8 = 56. Try and get the class to identify which property was relevant here (the distributive property) and write a sequence of equations capturing this reasoning $7*8 = 7*(5+3) = 7*5+7*3 = 35+21 = 56$ Another student might have used 7*2=14, 14*2=28, 28*2 = 56. This uses the associative property of multiplication: $7*8 = 7*(2*2*2) = ((7*2)*2)*2 = (14*2)*2=28*2 = 56$ ● The commutative property (4 + 8 = 8 + 4) is important not only in showing equality, but in many aspects of mathematics. An example is when students are learning their math facts. When students know that 7 + 4 = 11, they also know that 4 + 7 = 11.

		 ● The addition chart is a great way to help students see that they do not need to memorize a lot of facts, instead, they can visually see how to learn their addition facts in an easy way. Using a variety of different colors, the students can show different ways to recognize sums, using properties of arithmetic. ● Additive identity- when you add 0 to any number you get the same number you started with (students color both the row and column for +0. ● Some other columns and rows to color- +1, +2 (these are easy to do in your head) doubles (most students learn these quite early), doubles +1 and doubles +2. ● Once these have all been colored in, students can color in everything either above or below the double line, which allows them to see the commutative property.
	2e. Compare and contrast standard algorithms for operations on multi-digit whole numbers that rely on the use of place-value units (e.g., ones, tens, hundreds, etc.) with mental math methods students generate. *	● Four students show their work for 247 – 59. Are all four methods valid? Show a computation for this problem using a more traditional algorithm. How would each student compute 321 – 176 using their own method? Student 1 247 - 59 -2 -10 200 -2 190 188

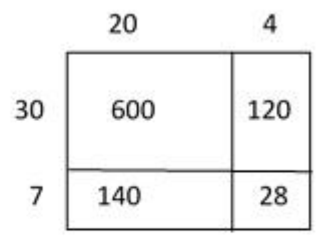
		Student 2: 247 minus 60 is 187, but I subtracted 1 too many, so 188. Student 3: 247 minus 47 is 200. I need to take away 12 more, so 188. Student 4: 247 plus 41 is 288, now I take away 100 to get 188.
	2f. Use math drawings and manipulative materials to reveal, discuss, and explain the rationale behind computation methods. *	● When drawings and manipulative materials are used, start with examples that are easily modeled and progress to more difficult examples. Examples used with elementary students should begin concrete, progress to pictorial, and finally progress to symbolic. Teacher candidates should revisit these experiences in this course first having hands-on experiences with base-ten blocks then drawing base-ten block, rectangular model, and other types of pictures to represent computations, and finally recording computations symbolically. Teacher candidates must understand that they are to teach conceptually rather than procedurally. That is, elementary students should understand what they are doing and why they are doing it rather than just following steps in an algorithm or procedure. ● The 3rd grade Ohio Standards includes the convention that a ´ b means a groups of b objects each. While by the Commutative Property it holds that a ´ b = b ´ a, it is helpful in teaching multiplication to maintain a consistent interpretation. Teacher candidates should understand and be prepared to teach strategies based on place value and the properties of operations. ● When both factors are single digits, it is typical to draw all the boxes in a rectangular model diagram. When dealing with two-digit factors, place value and the distributive property can be used. Example: 3 x 4 is usually represented as  4 x 3 is usually represented as



3 x 24 can be represented as

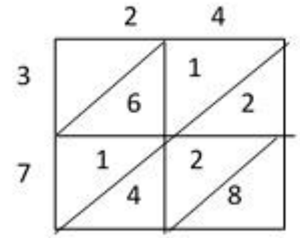


37 x 24 can be represented as



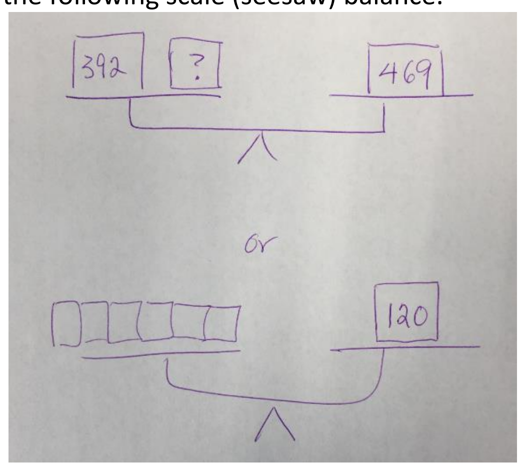
Notes: Models are not always drawn to scale. There are free software (such as GeoGebra) where examples such as these can be generated.

- Students should be exposed to a variety of models such as the lattice model:

		 ● Area models should also be used to illustrate division. ● Students should be exposed to both long and short form symbolic computations. ● Begin with two sets of bundles sticks representing the addends and then regroup to "complete" bundles of 10. ● For subtraction rephrase in terms of addition, think of what bundles are missing
	2g. Extend algorithms and mental math methods to decimal arithmetic. *	● Connecting mental math and symbols (representing children's work in "number sentences" or equations) - 81 - 12 could be solved in any of the following ways: - 81 - 10 – 2 - 80 – 11 - 79 – 10 - 80 - 10 + 1 – 2 - (81+8) - (12 + 8) - Ask students to generate at least one additional method that is not already shown. - How do these methods work on another problem, such as 1000 - 999? - Could use same strategies with 8.1 - 1.2 ● If fraction computations are covered prior to decimal computations, then extending algorithms to decimal arithmetic can be justified using fraction and whole number models. With whole numbers ones are added to or subtracted from ones, tens are added to or subtracted from tens, and so on. Similarly, with fractions, it was necessary to have a common denominator to add or subtract because one must add or subtract pieces of the same type. With decimals, one must line up the decimal

		point so that tenths are added to or subtracted from tenths, hundredths are added to or subtracted from hundredths, and so on. ● When two decimals are multiplied, the rule of counting the total number of decimal places in the two factors to place the decimal point in the product can be justified by using the word form of the decimals to convert both factors to fractions, multiplying, and then reasoning about the placement of the decimal point based on the denominator (which should be a power of ten). ● When a decimal number is divided by a decimal number, the convention of moving the decimal place in the divisor to create a whole number divisor and moving the decimal place in the dividend the same number of places is justified by converting the two decimal numbers to a fraction and considering an equivalent fraction.
	2h. Use different representations of the same fraction (e.g., area models, tape diagrams) to explain procedures for adding, subtracting, multiplying, and dividing fractions. (This includes connections to grades 6–8 mathematics.). *	● Many different fraction models may be used to illustrate fraction computations. It is essential that teacher candidates are comfortable with a variety of models. For addition and subtraction, models nicely illustrate how $\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b}$. Teacher candidates may remember being told this as a rule, but they will be excited to see how models illustrate the sense of this rule. The need for a common denominator for addition and/or subtraction can be motivated by the manipulative materials and models used. $$\frac{3}{4}+\frac{2}{5}=\frac{15}{20}+\frac{8}{20}=\frac{23}{20}$$ ● For multiplication use the idea that m x n represents m groups with n objects in each group to extend multiplication ideas to fractions. Initial examples may be presented where one of m or n are whole numbers and the other is a fraction. Pizza (with rectangular pies resembling tape diagrams or fraction pieces) and/or area models can be used to illustrate that $\frac{a}{b}$x $\frac{c}{d}=\frac{ac}{bd}$ without the need to convert the two factors to equivalent fractions having a common denominator prior to the multiplication being performed.

		• Division of fractions can also be presented conceptually using real-world examples and situations. Initial examples could involve dividing a fraction by a whole number or a whole number by a fraction with later examples involving division of two fractions. Models can be used to illustrate that when two fractions have the same denominator, then the result of dividing the fractions is equivalent to the result of dividing the numerators. That is, $\frac{a}{b} \div \frac{c}{b} = \frac{a}{c}$. This result can be used to prove the more general result that to divide two fractions, you invert the divisor and multiply. That is, $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bd} \div \frac{bc}{bd} = \frac{ad}{bc} = \frac{a}{b} \times \frac{d}{c}$. However, it is important that teacher candidates understand that the common denominator found here is used to justify the division algorithm and that finding a common denominator is not necessary in dividing fractions. • In all fraction computations, students should be encouraged to use estimation and number sense prior to computations. When presented with a problem such as $\frac{4}{9} + \frac{2}{15} + \frac{5}{9}$ students should recognize that commutative and associative properties can be used to find the answer of $1\frac{2}{15}$ without ever finding a common denominator.
	2i. Explain the connection between fractions and division, a/b = a ÷ b, and how fractions, ratios, and rates are connected via unit rates. (This includes connections to grades 6–8 mathematics. See the Ratios and Proportional Relationships Progression for a discussion of unit	• A number line or string model can be used to illustrate the idea that $a \div b = \frac{a}{b}$. For example, if a string of length 1 foot is divided into 3 equal parts, each part is one third of a foot i.e. $1 \div 3 = \frac{1}{3}$. If a string of length 3 feet is divided into 4 equal parts, each part is ¾ of a foot, i.e. $3 \div 4 = \frac{3}{4}$. • Fractions, ratios, and rates are connected via “unit rates.” A *unit rate* is a rate or a ratio with 1 in the denominator. That is, how much of something per one unit of something else. For example, if 100 students walk over a campus bridge in a two-hour

	rate.). *	period, the unit rate is 50 students per hour. Note that the unit that an answer is to be expressed in is important in computing the unit rate. For example, if you can type 20 words in 30 seconds, the unit rate is either 2/3 of a word per second or 40 words per minute. The relationship of ratios and proportional relationships will be explored further in Course II.
	2j. Explain why the extensions of the operations to signed numbers make sense. *	● A poker chip model and/or a number line model can be used to illustrate and explain computations with signed numbers.
Algebraic Thinking	**3a.** Model and communicate their reasoning about quantities and the relationships between quantities using a variety of representations. *	● 347 = 3 hundreds + 4 tens + ___ ones = 2 hundreds + ____ tens + 7 ones = 2 hundreds + ____ tens + 17 ones ● Students can give symbolic answers to problems like the one above or they may draw pictures to illustrate or they may use physical manipulatives (e.g. base ten blocks) or online manipulatives to show their answers. ● This picture should represent a pan balance (or a seesaw) that is only balanced when the same amount of weight is placed on each side. Find the amount that will make the following scale (seesaw) balance.  ● Example: Looking at the pictures above, we can say that the top scale will balance if the left side is equal to 469, so we have to find a number to add to 392 that will give us 469. In equation form, this could be written as 392 + __ = 469. (It could also be

		interpreted as 469 - __ = 392). The second scale could be described with an equation like 6X = 120. ● Example: In the bottom picture, assume that the 6 empty boxes are of equal weight. How can you determine the weight of each box? ● Include other numbers for and connect the seesaw movement with inequalities.
	3b. Discuss the foundations of algebra in elementary mathematics, including understanding the equal sign as meaning "is the same [amount] as" rather than a "calculate the answer" symbol. *	● Understand that an equation is an assertion that two quantities are equal. Such an assertion can be true or false. ○ 3 x 5 = 15 is a true equation ○ 3 x 5 = 5 + 10 is a true equation ○ 3 x 5 = 10 + 10 is a false equation ○ "3 x __ = 15" could be true or false depending on what you put in the blank spot. Often, we ask children to find the number which would make the equation true. ● "Which of the following statements do you think children would accept as "equations" and which are not (if any): 3 x 5 = __, 3 x __ = 15, 15 = __ x 3, and 3 x 5 = 5 x 3? How about 3 x 5 = 20? Explain your thinking." ○ Children's interpretations are often with the idea of "=" as "and here comes the answer" or "now do something." However, an unknown may appear in any position of an equation. ○ In addition, students / teachers must realize that when multiple equal signs are used they in fact must all be equivalent expressions. ○ 3 x 5 = 20 is a false equation which will later lead to inequalities. Children learn about false equations and inequalities beginning in the first grade. ● Question: A teacher gives their students a prompt: which numbers should you fill in the blank to make the equations true? 2 + 6 = __ x 2 = __ What numbers should the students use in these blanks to make the equations true? What common mistakes do you think they might make, and why? ○ Answer: The first blank should be a 4, and the second blank should be an 8 to make all of the equations true.

<table>
<tr><td></td><td></td><td>Many students might fill in 8 for the first blank, and 16 for the second blank. They would do this, because children's interpretations are often with the idea of "=" as "and here comes the answer" or "now do something.", rather than the equal sign asserting the equality of the two sides. An unknown may appear in any position of an equation.

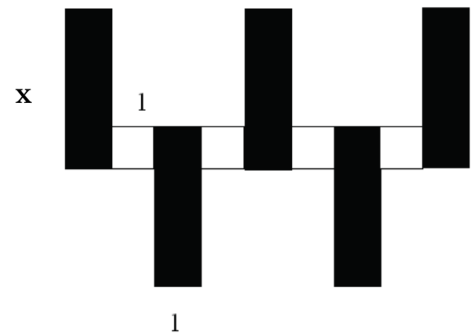
In addition students / teachers must realize that when multiple equal signs are used they in fact must all be equivalent expressions.

● Equal Shmequal: A Math Adventure by Virginia Kroll is a great way to represent equality through children's literature.</td></tr>
<tr><td></td><td>3c. Look for regularity in repeated reasoning, describe the regularity in words, and represent it using diagrams and symbols and communicate the connections among these. *</td><td>● https://www.nctm.org/Publications/Mathematics-Teacher/2017/Vol110/Issue7/What-is-%E2%80%9CRepeated-Reasoning%E2%80%9D-in-MP-8_/
○ The above reference clarifies the phrase "regularity in repeated reasoning", but the examples given are mostly too high level to be appropriate for this course.

Examples of this standard which are appropriate for this course:
● Example: A shipping service has a minimum charge of $8.99 for shipping your first package. It costs $3 for each additional package that you want to ship on the same day. Explain how you would figure out the cost to ship N packages on one day. Write an expression for your method of figuring this out. Answers: I would pay $8.99 for the first package and add on $3 more for each of the remaining packages. I could write this total charge using the expression 8.99 + 3 (N-1).
● Any time we generalize from particular instances to general formulas is an example of recognizing regularity in repeated reasoning. For instance, we might multiply many different fractions using an area model to guide us, recognize the repeated reasoning at work, and then use symbols to capture the general rule for multiplication of fractions.
● https://www.pleacher.com/mp/puzzles/tricks/nums.html</td></tr>
</table>

		• Students can find many interesting patterns in hundreds charts and addition and multiplication tables.
	3d. Articulate, justify, identify, and use properties of operations. *	• Properties of Addition: 0 identity, commutative, associative. • Properties of Multiplication: 1 identity, associative, commutative, distributive (multiplication over addition). • Students should be able to know what the property is called and what it means, give examples that demonstrate the property, and explain why this property holds true • Use symbols to describe a property. • One way to show the commutative property of addition is with a number line. • Number talks are a good way to address many of the properties. An example is 12 X 15. The students solve this in any way they choose and then have a class discussion and represent each of the students' responses to show the different properties. Example: 12*15 = 12*(10+5) = 12*10 + 12*5 = 12*10+12*10/2 =120+120/2 = 120+60 =180

	3e. Describe numerical and algebraic expressions in words, parsing them into their component parts, and interpreting the components in terms of a context. *	• Recognize that 3 × (18,932 + 921) is three times as large as 18,932 + 921, without having to calculate the indicated sum or product. (5.OA.2).
		
		• Alan found four marbles to add to his 5 marbles currently in his pocket. He then had a competition with his friends and tripled his marbles. Write a numerical expression to model this situation without performing any operations (learning.com).
		• Common types of test questions:
		  ○ Fifteen less than twice a number
		  ○ Three times a number, increased by seventeen
		  ○ The product of nine and a number, decreased by six
		
		• A task from Jo Boaler's paper here https://ed.stanford.edu/sites/default/files/boaler__staples_2008_tcr.pdf

		is a good example. Ask students to find an expression for the perimeter of this shape:  They might want to first try using particular values for x, then recognize the regularity in repeated reasoning and develop an explicit formula for the perimeter including x (10x+10 is such a formula). A quite challenging question to ask (interpreting the components in terms of a context) is "Where is the coefficient of 10 in the picture? Where is the +10 in the picture?"
	3f. Use a variety of methods (such as guess and check, pan balances, strip diagrams, and properties of operations) to solve equations that arise in "real-world" contexts. *	● A word problem like "Alphys has 5 more oranges than Undyne. In total, they have 45 oranges. How many oranges do they each have?" could be solved by ○ Guessing and checking (try several pairs where one is 5 more than the other, adjust to get closer). ○ Pan balances (set up the equation U+(U+5) = 45 and solve graphically using a pan balance diagram). ○ Strip diagrams (Set up equation and solve graphically using strip diagrams).
Number Theory	**4a.** Demonstrate knowledge of prime and composite numbers, divisibility rules, least common multiple, greatest common factor, and the	● Be able to give definitions and examples of the words "factor, multiple, greatest common factor, least common multiple, prime, composite". ● Relate prime/composite to a rectangular area model: can I build a rectangle with a given integer area using integer side lengths in more than one way? ● Explain divisibility rules (such as the divisibility rule for 3) by thinking through base 10 structure and remainders.

	uniqueness (up to order) of prime factorization. *	● Compute LCM and GCF using a variety of methods and explain why those methods are valid. ● State and use the uniqueness of prime factorization. For instance, explaining why 11*37 = 7*41 cannot be a true equation without actually computing both sides.
	4b. Discuss decimal representation and recognize that there are numbers beyond integers and rational numbers. *	● Use a division algorithm to find decimal expansion of a rational number. For instance, express 5/7 as 0.714285714285… ● Argue informally that the decimal expansion of a rational number must repeat or terminate. ● Use that observation to construct real numbers which are not rational (such as 0.101001000100001...).