TMM022: Mathematics in Elementary Education II (Illustrations)

Typical Range: 4-5 Semester Hours Recommendation:

It is essential for all teachers of mathematics to understand the reasoning underlying the mathematics they are teaching. They need to understand why various procedures work, how each idea they will be teaching connects with other important ideas in mathematics, and how these ideas develop and become more sophisticated. Furthermore, knowing only the mathematics of the elementary grades is not sufficient to be an effective teacher of elementary grades mathematics. Neither is it sufficient to require that future teachers simply "take more math courses." This document describes the kinds of mathematics and mathematical experiences that we believe are essential for their mathematical learning and professional development. Exploration in middle grade topics are introduced at the elementary level to begin planting the seed in preparation for advanced teachings.

We take the view that mathematics courses for future teachers must prepare them to do a different kind of work in their mathematics teaching than they likely experienced in their own schooling. With this in mind, we encourage oral and written communication in these classes as both a learning tool and as preparation for handling mathematical questions which arise in their classrooms. For example, we aim for discussion that focuses on the deep mathematical reasoning underlying the computational procedures that are usually taught in elementary school. We recommend this be done by exploring common misconceptions with preservice teachers and by engaging future teachers in activities that require them to interpret their own multiple ways of addressing questions and interpreting children's work which might be incorrect, incomplete, or different from their adult ways of thinking. We aim to encourage a serious approach to the mathematics through questions (by instructors *and* preservice teachers) about *why* we do things the way we do, what the operations *mean*, what the *units* are on the answers, and how the mathematical ideas of the day *connect* to other mathematical ideas (looking for the "big ideas" in a problem).

We recommend these courses be activity based so that opportunities for deep, connected learning arise while misconceptions are addressed. This requires "good" problems and "good" questioning by instructors. A good problem needs to engage the preservice teachers at an appropriate level of challenge (hard enough that the preservice teacher cannot answer on autopilot). Often this is accomplished by confronting them with misconceptions framed as "a child said this…" and directions to analyze and/or justify the result. The justifications can themselves become a source of deep discussion - one preservice teacher may not understand another's solution or explanation, a preservice teacher's correct answer may have a flawed explanation, or a new method may be generated once an array of other methods has been shared. Sometimes the discussion is generated when an instructor says, "I want to list all of the different answers we got before we discuss the reasoning" (which means the class can actually discuss their own wrong answers). All of the above takes a lot of time, so we recommend 8 to 10 credit hours of such work in teacher education programs or prerequisites.

At the same time, we strongly urge that instructors be aware that these are college level mathematics courses. "A credit-bearing, college-level course in Mathematics must use the standards required for high school graduation by the State of Ohio as a basis and must do at least one of the following: 1) broaden, or 2) deepen, or 3) extend the student's learning." We recommend dedicated coursework for this content because there is much deep mathematics to be explored in understanding what we loosely term "elementary school mathematics." This is a mathematical content course. The learning outcomes are all focused on using, justifying, and connecting mathematical concepts and do not address "how to teach." It is sometimes appropriate to discuss topics which are more directly relevant to a methods course when they serve the purpose of motivating a mathematical discussion, but students should not be assessed on methods in these courses.

The courses should integrate reasoning, flexibility, multiple explanations, and number sense. Leading questions help students to make connections among topics and to develop their own questioning skills (e.g. Have we seen this idea before? How are these two different solution methods related to each other? What does it mean? How do you know? Could you draw a picture to show it? Where did they go wrong? Could we use their idea in this other problem?) Students should understand that mathematics is correct if it makes personal sense and if one can explain it in a way to make sense to others, not if an authority certifies it. Preservice teachers should leave these courses knowing that math makes sense and armed with the underlying knowledge they need to make math make sense for their future students. Elementary students will often come up with their own creative approaches to problems, so future teachers must be able to evaluate their mathematical viability *before* deciding how to respond instructionally. Squashing a child's idea can be quite harmful. And often children's ideas are right or almost right.

The learning outcomes that follow are all viewed as essential by the committee (marked with an asterisk) and demonstrate the level of student engagement motivating this course. In addition, learning outcomes are not specific items/topics but rather learning outcomes of course entirety. Institutional courses should provide an integrated experience with learning outcomes woven together throughout the course.

<table>
<tr><th colspan="2">Learning Outcomes</th><th>Illustrations</th></tr>
<tr><td>● Ratios, Proportional Relationships, and Functions (this has connections to Grades 6-8)</td><td>1a. Reason about how quantities vary together in a proportional relationship, using tables, double number lines, and tape diagrams as supports. *</td><td>● A lemonade recipe uses 2 cups of lemon juice and 5 cups of water.
○ Make a table showing how much water to add to a given amount of lemonade. Use the table to reason about how much water would be needed for 3 cups of lemon juice
○ Do the same problem as above but using a double number line.
○ Use a tape diagram to answer the question "How many gallons of lemon juice and water would be needed to make 10 gallons of lemonade?" (Note the change in units here).</td></tr>
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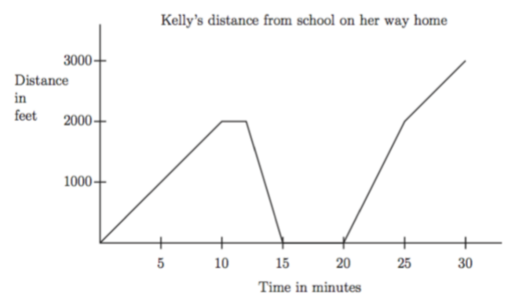
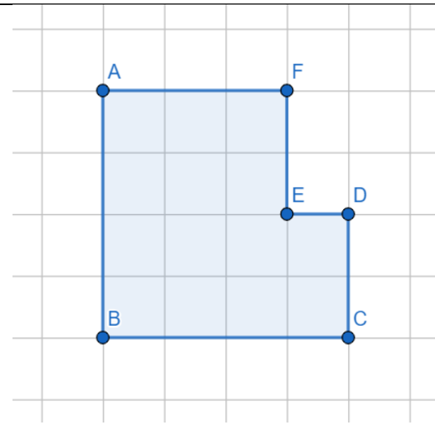
	1b. Distinguish proportional relationships from other relationships, such as additive relationships and inversely proportional relationships. *	● A shade of green paint is made by combining 3 liters of blue paint and 2 liters of yellow paint. ○ Billy has 5 liters of blue paint and wants to figure out how much yellow paint to add to make the shade of green. He reasons that since he has two more liters of blue paint, he will also need two more liters of yellow. He concludes that he needs 4 liters of yellow paint. Is this reasoning correct? If not, can you explain why this reasoning is incorrect? ● The area of a rectangle is 12 square inches. Is the relationship between the length and width additive, proportional, inversely proportional, or none of these? Explain your answer. ● Come up with your own example of an inversely proportional relationship. Explain why it is inversely proportional. ● Can a relationship be both additive and proportional?
	1c. Use unit rates to solve problems and to formulate equations for proportional relationships (see measurement). *	● You worked 40 hours this week and made 360 dollars. ○ How much money would you make if you worked 11 hours? (Note: student could compute the unit rate of $9/hour to help them with this problem). ○ You lost your records of how many hours you worked last week, but you can see in your bank account record that you got $288. How many hours did you work? (Note: student could compute the unit rate 1/9 hours/dollar to help them with this problem.) ○ Let the letter H stand for the number of hours worked, and M stand for how much money you earned. Write three correct equations relating these quantities and explain why they are true. (Note: possibilities include M=9H, H =(1/9)M, 360H = 40H. First two use unit rates).
	1d. Recognize that unit rates make connections with prior learning by connecting ratios	● Consider the lemonade example from 1a. Name two different unit rates associated with this ratio. Is a ratio the same thing as a fraction? ○ Monique wants to find out how much lemon juice to add to 16 cups of water. She sets up the equation ⅖ = J/16. Is this equation correct? If so, explain why using the concept of a unit rate. If not, use unit rates to find a correct equation.

	to fractions. *	
	1e. View the concept of proportional relationship as an intellectual precursor and key example of a linear relationship. *	● Todd started with 3 stones in his pocket. Each day, he added 4 more stones. Model the relationship between the number of days which have passed and the number of stones in Todd's pocket as an algebraic equation. Is this a linear relationship? Is it a proportional relationship? Explain your reasoning (make sure to refer to the definition of linear and proportional relationship in your answer).
	1f. Examine and reason about functional relationships represented using tables, graphs, equations, and descriptions of functions in words. In particular, students can examine the way two quantities change together using a table, graph, and equation. *	● A regular cheeseburger (with one patty) costs $3. Each additional patty costs $1.25. Represent the relationship between the cost of a burger (C) and the number of patties on that burger (P) using a table, a graph, and an equation. ● The graph below shows Kelly's distance from school while she is walking home. Write a story about Kelly's walk home that fits with this graph. Discuss how your story fits with specific parts of the graph. Be sure to include all features of the graph in your story.  ○ How fast was Kelly walking at time t=5? Explain how to think of this as a unit rate. Explain how this is connected to the slope of the graph.
	1g. Examine the patterns of change in proportional and linear relationships and the types of	● These kinds of real-world modeling situations are reflected in the examples above.

	real-world situations these functions can model and contrast with nonlinear relationships. *	
● **Measurement**	**2a.** Explain the general principles of measurement, the process of iterations, and the central role of units (including nonstandard, U.S. customary, and metric units). *	● Choice of measurable attribute ○ A chalkboard eraser has the following measurable attributes: length of one edge, the area of one of its faces, its weight, its volume. ○ The chalkboard also has all of these attributes. ● Choice of unit ○ Size of unit affecting measurement If you instead stack these erasers with their longest side pointing up, then we are using a different unit to measure the same height. We need fewer units than we did before, because these units are longer. ● Iteration, additivity, and invariance ○ By "iteration" we mean that we are using the unit to measure our object without any gaps or overlaps. ○ An example of "additivity" is measuring a building and its lightning rod by first measuring the building with some unit (a common one like "feet" or an uncommon one like "Spiderman heights") then measuring the height of the lightning rod using the same unit, and recognizing that one can find the total height by summing the two measurements. However, note that if we decompose a rectangle into two rectangles the perimeter is NOT additive. ○ By "invariance" we mean invariance under certain transformations (such as length, area, and volume being invariant under translation, but not dilation)

		- Could also explore time (metronome tick to measure time, or clock tick, etc.). To see how long in time an event is, we need a repeatable duration (like metronome tick) and see how many of those durations fit into the event whose duration we are trying to measure. - Capacity: to measure the volume (volume and capacity are the same) of a bottle of detergent in capfuls, we would keep filling capfuls and counting how many we get until we run out. A standardized unit of volume like a “cup” is similar but has an agreed upon consistent meaning across our culture. - Could also do weight and temperature (although temperature is hard to determine what you are actually measuring...) - Money is one of the measurements of the worth of an object or service. We can measure the worth of some things (like candy bars or houses) using different units of money (such as dollars, pennies, etc.). ● Could do an activity which involves comparing the areas of two irregular shapes. How can you tell which has more area? Allow students to come up with their own organic explanations and compare/contrast different solutions. - To measure area, you need to choose a unit which has area. For instance, you can try to estimate how many index cards it takes to cover your desk. Do your index cards all need to be oriented all in the same direction? Orientation of the index card, and the length of the two sides matters a lot. This is NOT length times width but leads to that idea. 3 inch by 5 inch rectangle is 3 rows of 5 square inches. ● Conduct estimation scavenger hunts with both standardized and non-standardized units of measurement. For example, find an object that is x paper clips, inches, centimeters in length.

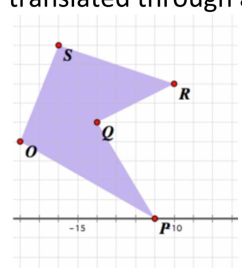
		■ Use non-linear (crooked) length and have students measure the length of a crooked distance using standardized and non-standardized units. ■ Van De Walle has 20+ instructional activities on length for grades K-5 than can be used with pre-service teachers and students. OR this document from progressions documents Univ. of Arizona. https://commoncoretools.files.wordpress.com/2011/06/ccss_progression_md_k5_2011_06_20.pdf ● Explore children's mistakes with measuring length (counting tick marks instead of spaces)' ● Be precise with academic language: a 3 by 5 index card has area 15 in^2. This should be read as "15 square inches" not "15 inches squared." ● Children may not realize that base ten is not an appropriate system to use to compute elapsed time (e.g. *It is now 1:45, school ends at 3:00. How much longer is the school day?* May be incorrectly answered as below. 3:00 - 1:45 1:55 ● Have students measure their desks using their pencil, then measure their pencil using a pen cap to estimate how long, in pen caps, their desk is. ● Using a cellphone case edge to measure a length brings up the important point that you have to choose which edge to use. ● Children sometimes count perimeter of a shape by counting squares around, which leads to "only counting the corners once". ● The geometry application at geogebra.org can be used to draw grid figures for area and perimeter measurement.

		 ● http://toytheater.com/category/teacher-tools/ has online Geoboards and other measurement manipulatives.
	2b. Explain how the number line connects measurement with number through length. *	● The numbers are not really labels for the points, they are labels for the number of units to the origin. ● A common misconception is counting the number of tick marks rather than the number of unit intervals. ● In the CCSS and in Ohio's 2017 Learning Standards: Mathematics there is a progression of the use of line plots to display measurement (i.e. 2nd grade - the horizontal scale is marked in whole number units; 3rd and 4th grades - the horizontal scale includes fractions appropriate for the grade levels; 5th grade includes both fractions and decimals).
	2c. Understand and distinguish area and volume, giving rationales for area and volume formulas that can be obtained by finitely many compositions and decompositions of unit squares or unit cubes,	● Develop the formula for the area of a rectangle in terms of counting the number of square units present (potentially also non-square units!). See a 4 in by 5 in rectangle as containing 4 rows, with 5 square inches in each row. ● Sometimes you don't have enough information for a formula. For instance, the area of a parallelogram cannot be determined from its side lengths. ● Several different arguments for triangles: if the height is over the base, you can see the triangle as half of a rectangle. If the height is not over the base, you need to do some algebra.

	including but not limited to formulas for the areas of rectangles, triangles, and parallelograms, and volumes of arbitrary right prisms. (This includes connections to grades 6–8 geometry, see the Geometric Measurement Progression.). *	● Parallelograms can be decomposed into 2 triangles, or sometimes cut/pasted into rectangles (not always depending on "how oblique" the parallelogram is). ● Shearing ideas are optional. ● See the volume of a prism in terms of counting numbers of unit cubes. ● Area of a circle via decomposing the circle into "pizza slices" and rearranging into an approximate rectangle. Approximation gets better the more slices you take. ● Optional: finding areas of sectors of circles can reinforce fraction ideas. ● Volumes of cones are optional (argument via cavalieri's principle is quite hard. Can "justify" with physical models). ● Surface area of cone also optional, but it does tie many ideas together (area of circle, arclength, area of sectors, etc.).
	2d. Describe how length, area, and volume of figures change under scaling, focusing on areas of parallelograms and triangles, with counting-number scale factors. *	● Reason about scaling in several ways: If an 18-inch by 72-inch rectangular banner is scaled down so that the 18-inch side becomes 6 inches, then what should the length of the adjacent sides become? Explain how to reason by: ▪ Comparing the 18-inch and 6-inch sides. [The 18-inch side is 3 times the length of the 6-inch side, so the same relationship applies with the 72-inch side and the unknown side length.]. ▪ Comparing the 18-inch and 72-inch sides. [The 72-inch side is 4 times the length of the 18-inch side, so the unknown side length is also 4 times the length of the 6-inch side.]. ● The behavior of surface area and volume under scaling. ○ Understanding that if you scale the lengths in a figure by a factor of k, then the surface area will scale by a factor of k^2, and the volume will scale by a factor of k^3. ○ This can be explained by thinking about scaling up each 1 unit by 1 unit square on the surface to a k unit by k unit square, and each cubic unit to a k unit by k unit by k unit cube.

		○ Also helpful for thinking about which attributes of a shape are related to different dimensions. For instance, the weight of two objects of constant density is proportional to their volume.
	2e. Informally develop the formulas for area and circumference of a circle and use them in solving real-world problems. *	● Cut a circle into "pie" pieces. Stack them opposite each other to create an approximate parallelogram where the "height" of the parallelogram is the radius of the circle and the base is approximately ½(2πr). A= r (πr) or πr^2.
	2f. Attend to precision in measurement with rounding guided by the context. *	● The context here should be measurement. What is the most appropriate measure for the width of a door? Explain your reasoning. Which would you choose for the measure of the weight of gold? 1lb. or 1.023 lbs.? Explain your answer.
	2g. Convert between different units both by reasoning about the meaning of multiplication and division and through dimensional analysis. *	● Give problems such as how many centimeters are there in 2.5 meters or How many kilometers are there in 340 centimeters? Ask how you find out how many inches are in 1 mile or How many seconds you have been alive as of today at the same time you were born.
● **Geometry**	**3a.** Understand geometric concepts of angle, parallel, and perpendicular, and use them in describing and defining shapes. *	● The way we measure angles should be compared to other measurements we have made. ● Measuring angles in degrees (one choice of unit) vs. turns (another choice of unit) ● Laying your unit angles "ray to ray" while matching vertices. ● *Revisit angle when discussing similarity and scaling* ● Parallel postulate (ties into theorem/definition and axiom discussion below). ■ Different ways to think about parallel ● Distance between two lines is always the same (railroad tracks). ● Two roads are parallel if they are going in the "same direction". ○ (Aichele? and Wolf(e?)).
	3b. Describe and reason about spatial locations (including the coordinate	● Understanding the role of definitions, and separating definitions from the consequences of the definitions. For example, a triangle could be defined to be "a polygon with three sides", but it would be less than optimal to

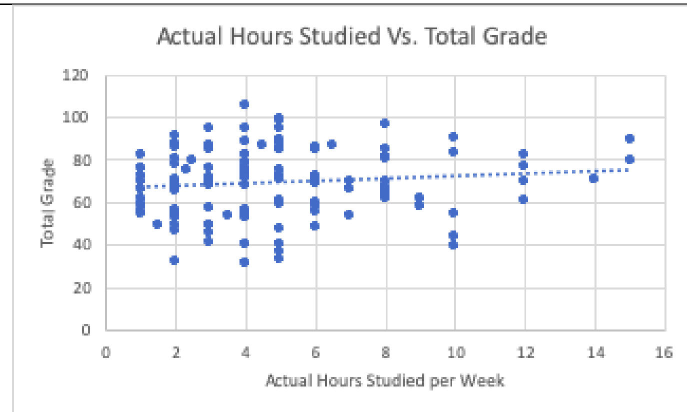
	plane). *	define a triangle as “a polygon with three sides whose angles sum to 180 degrees”, because the angle sum theorem for triangles can be proven for any 3 sided polygon. ● An idea would be to give a wide variety of unusual figures and attempt to come up with good definitions which capture various features of those shapes. For instance, it would be interesting to see if students could develop a workable definition for convexity. ● Given 3 vertices in the coordinate plane of a specified type of quadrilateral (e.g. parallelogram), students should be able to locate the 4th vertex.
	3c. Informally prove and explain theorems about angles and solve problems about angle relationships. *	● MP 7 Look for and make use of structure. For example, use some previously shown theorems of parallel lines and supplemental angles to show why the sum of the measures of the 3 angles of a triangle is equal to 180 degrees. ● Vertical angles are congruent. ● Find the measure of the angle between the hands on an analog clock at the time 9:30 (the answer is not 90 degrees).
	3d. Classify shapes into categories and reason to explain relationships among the categories. *	● Reason about scaling in several ways: If an 18-inch by 72-inch rectangular banner is scaled down so that the 18-inch side becomes 6 inches, then what should the length of the adjacent sides become? Explain how to reason by: ○ Comparing the 18-inch and 6-inch sides. [The 18-inch side is 3 times the length of the 6-inch side, so the same relationship applies with the 72-inch side and the unknown side length.] ○ Comparing the 18-inch and 72-inch sides. [The 72-inch side is 4 times the length of the 18-inch side, so the unknown side length is also 4 times the length of the 6-inch side.] ▪ MP 2 Reason abstractly and quantitatively. ▪ MP 4 Model with mathematics. ▪ MP 7 Look for and make use of structure. ● The behavior of surface area and volume under scaling.

		○ Understanding that if you scale the lengths in a figure by a factor of k, then the surface area will scale by a factor of k^2, and the volume will scale by a factor of k^3. ○ This can be explained by thinking about scaling up each 1 unit by 1 unit square on the surface to a k unit by k unit square, and each cubic unit to a k unit by k unit by k unit cube. ○ Also helpful for thinking about which attributes of a shape are related to different dimensions. For instance, the weight of two objects of constant density is proportional to their volume.
	3e. Explain when and why the Pythagorean Theorem is valid and use the Pythagorean Theorem in a variety of contexts. *	● One expects some background with area for some examples of the Pythagorean theorem. Building squares on each of the sides of a triangle explore when the areas of the two smaller squares are the same as the area of the largest square. Also note what happens to the areas of the squares when the triangle is NOT right. ● Relate the area of a square to a number squared (part of the formula).
	3f. Examine, predict, and identify translations, rotations, reflections, and dilations, and combinations of these. *	● Given the shape below. Describe in some detail how the shape would change if it were rotated about point P 90 degrees. Describe in some detail how would this shape change if it were reflected in the horizontal line shown. Describe in some detail how the shape would change if it were translated through a vector (3, -2).  If the shape above were reflected about the horizontal line shown and then through a vertical line through P describe the result of these reflections on this shape. ● The Geometry App at www.geogebra.org can be used to illustrate translations, rotations, reflections, and dilations.
	3g. Understand congruence in terms of	● If a shape can be mapped onto a second shape with a series of translations, reflections or rotations, then it is congruent.

	translations, rotations, and reflections; and similarity in terms of translations, rotations, reflections, and dilations and solve problems involving congruence and similarity. *	● If a shape can be mapped onto a second shape with a series of translations, rotations, or reflections, and a dilation, then the shapes are similar. Mr. Khan's video projector is projecting his shape onto a screen for his class to see. What transformation could we use to show that the two shapes (actual and projected shape) are similar shapes?
	3h. Understand symmetry as transformations that map a figure onto itself. *	● Given a square and the intersection of its diagonals. Describe the rotational symmetries; which lines of reflection map the square on itself. After identifying these symmetries of a square investigate various products of these symmetries to see that they produce one of the 8 symmetries.
● **Statistics and Probability**	**4a.** Recognize and formulate a statistical question as one that anticipates variability and can be answered with data. *	● Which of the following questions are statistical questions and which are not? ○ How many minutes in a leap year? ○ How many minutes do 5th graders work on homework in a week? ○ How is the number of minutes a 5th grader works on homework associated with their exam scores? ● Use reasoning about proportional relationships to argue informally from a sample to a population. For example: ■ If 10 tiles were chosen randomly from a bin of 200 tiles (e.g., by selecting the tiles while blindfolded), and if 7 of the tiles were yellow, then about how many yellow tiles should there be in the bin? Imagine repeatedly taking out 10 tiles until a total of 200 tiles is reached. What does this experiment suggest? Then investigate the behavior of sample proportion by taking random samples of 10 from a bin of 200 tiles, 140 of which are yellow (replacing the 10 tiles each time). Plot the fraction of yellow tiles on a dot plot or line plot and discuss the plot. How might the plot

		be different if the sample size was 5? 20? Try these different sample sizes.
	4b. Understand various ways to summarize, describe, and compare distributions of numerical data in terms of shape, center (e.g., mean, median), and spread (e.g., range, interquartile range). *	● Students use the question(s) selected (i.e. How many hours a week do my classmates spend watching television/playing video games?). ● Summarize the data, decide which Measure of Central Tendency best represents the answer to the question? (mean, median, mode). ● Represent your findings graphically. ● Online or handheld graphing calculators can be used to create and compare boxplots.
	4c. Use measures and data displays to ask and answer questions about data and to compare data sets. (This includes connections to grades 6–8 statistics.). *	● Introduce the GAISE model 1. Formulate Question 2. Collect Data to Answer the Question 3. Analyze the Data 4. Interpret Results ● Students collect data to answer a statistical question (Data can also be obtained online if time is a factor). Suggested question: How many hours a week do my classmates spend watching television/playing video games?
	4d. Distinguish categorical from numerical data and select appropriate data displays. *	● Choose a set of categorical data and represent your results in a bar chart or pie graph. ● Choose a set of numerical data and represent your data in a line graph or scatter plot. ● Examine the distinction between categorical and numerical data and reason about data displays. For example:

		■ Given a bar graph displaying categorical data, could we use the mean of the frequencies of the categories to summarize the data? [No, this is not likely to be useful.] ■ Given a dot plot displaying numerical data, can we calculate the mean by adding the frequencies and dividing by the number of dots? [No, this is like the previous error.]
	4e. Use reasoning about proportional relationships to argue informally from a sample to a population. *	● "You call 100 random numbers from the phone book for your area and ask if they have gone to the doctor this month. 13 of them reply that they have been to the doctor, and the others say that they have not. If your area has 25000 total residents, estimate how many have been to the doctor this month. Additionally: what are the limitations of the way you conducted this survey? "
	4f. Calculate theoretical and experimental probabilities of simple and compound events and understand why their values may differ for a given event in a particular experimental situation. *	● "A game is played by flipping a coin and rolling two dice. You "win" if you get heads on the coin and roll a 5 or higher on either die. What is the theoretical probability of winning the game? Say you played this game 1000 times. Does your theoretical probably allow you to say exactly how many times you will win? If not, what information does the theoretical probability give you about this situation?"
	4g. Explore relationships between two variables by studying patterns in bivariate data. *	● "Here is a scatterplot showing the relationship between the number of hours a student studied for an exam, and their score on that exam." (from Le-Quan Ngo – Le-Quan Ngo)

		 “Is there a relationship between these two quantities? Would you say the correlation is weak or strong? Identify some interesting data points and explain why you think they are interesting.”