TMM015 – TECHNICAL MATHEMATICS II
(updated May 6, 2020)

Typical Range: 4-5 Semester Hours

A course in Technical Mathematics specializes in the application of mathematics to the engineering technologies. The course emphasizes critical thinking by placing students in problem-solving situations and supporting students as they learn to make decisions, carry out plans, and judge results. Students encounter contextualized situations where concepts and skills associated with measurement, algebra, geometry, trigonometry, and vectors are the pertinent tools. The course highlights the supporting algebraic and analytical skills.

As a mathematics course in the applied fields, students studying Technical Mathematics 2 (TMM015) will benefit from more active and collaborative learning. Instead of extensive lectures dominating the presentation of skills and procedures, we hope this course will place students in situations where the mathematics become the active tools for investigation. Applications should be the foundation of a collaborative experience, where groups of students make decisions, choose tools, follow plans, draw conclusions, and explain their reasoning.

To qualify for TMM015 (Technical Mathematics 2), a course must achieve all of the following essential learning outcomes listed in this document (marked with an asterisk). The Illustrations exemplify the level of student engagement motivating this course.

1. Basic Elementary Functions: Successful Technical Mathematics students recognize basic function forms, algebraically and graphically, and can predict general behavior from this classification. Students can produce graphs of basic functions and provide descriptions of their behavior. Finally, students can develop and present this analysis as they deem useful.

The successful Technical Mathematics student can:

1a. analyze basic polynomial functions. From a factored form, students identify zeros, multiplicities, locate and classify corresponding graph intercepts. From the graph or factorization, students discern locations of possible local maximums and minimums. Students produce graphs of polynomials given in factored form and describe the connection between zeros, factors, and intercepts. By examining the factored form, students can predict polynomial end-behavior. *

Illustrations:

- Given $p(x) = (x + 3)(x + 1)^2(x - 5)$, the student identifies zeros of p and corresponding intercepts on the graph of p.
- The student explains why the graph of $y = p(x)$ crosses or does not cross the x-axis at the corresponding intercept.
- The student explains how you know whether the graph of $y = p(x)$ will have a local maximum or minimum between the zeros.
- The student sketches a graph of $y = p(x)$.
- From the graph of $y = p(x)$, the student conjectures a possible formula for $p(x)$.

- The student recognizes $y = 2(x-2)^3 + 1$ as being related to $y = x^3$ via transformations and exploits this in order to sketch the graph.
- Given the graph of what appears to be a parabola, the student uses knowledge of the vertex and intercepts to create a formula for the function in the form $f(x) = a(x-h)^2 + k$ or $f(x) = ax^2 + bx + c$ or $f(x) = a(x-r_1)(x-r_2)$.
- Given the model $h(t) = 640t - 32t^2$ gives height in feet of a model rocket off the ground t seconds after liftoff, the student determines that a reasonable applied domain for the model can be obtained by solving $h(t) > 0$.

1b. analyze basic rational functions. From a factored form, students identify zeros, poles, multiplicities, locate and classify corresponding graph intercepts and vertical asymptotes. From the graph or factorization, students discern locations of possible local maximums and minimums. Students produce graphs of rational functions given in factored form and describe the connection between zeros, factors, and intercepts and asymptotes. By examining the factored form, students can predict polynomial end-behavior and represent this graphically. *

Illustrations:

- Given $R(x) = \frac{(x+3)(x+1)^2(x-5)}{(x+4)(x-3)}$, the student identifies zeros and poles of p along with their multiplicities.
- The student explains why the graph of $y = R(x)$ crosses or does not cross the x-axis at the corresponding intercept.
- The student explains whether or not the function $R(x)$ changes sign across the poles.
- The student identifies the domain intervals where R increases and decreases.
- From the graph of $y = R(x)$, the student estimates local maximums and minimums.
- The student provides domain intervals upon which the function is continuous.

1c. analyze basic exponential functions. Students recognize basic exponential forms from formulas and graphs. Students explain general function properties and behavior and produce graphs from these. *

Illustrations:

- Given $A(t) = 4e^{t-2} - 3$ or $A(t) = 4\left(\frac{1}{2}\right)^{t-2} - 3$ the student determines all zeros of A.
- The student identifies all asymptotes of the graph of $y = A(t)$.
- The student identifies domain intervals where A increases or decreases.
- The student sketches a graph of $y = A(t)$.
- The student provides domain intervals upon which the function is continuous.

1d. analyze basic logarithmic functions. Students recognize basic logarithmic forms from formulas and graphs. Students explain general function properties and behavior and produce graphs from these. *

Illustrations:

Given $B(t) = 4\ln(5t-1) - 2$ or $B(t) = 4\,log_2(5t-1) - 2$, the student determines all zeros of B.

- The student identifies all asymptotes of the graph of $y = B(t)$.
- The student identifies domain intervals where B increases or decreases.
- The student sketches a graph of $y = B(t)$.
- To solve $log(x) + log(x-2) = 1$, the student solves $log(x^2 - 2x) = 1$ and knows this procedure may result in extraneous solutions.
- The student can explain what an extraneous solution is.
- The student provides domain intervals upon which the function is continuous.

1e. analyze basic roots and radical functions. Students describe general function properties and behavior and produce graphs from these. *

Illustrations:

- Given $f(x) = 4\sqrt{3-x} - 1$, the student determines all zeros of f.
- The student determines the domain of $f(x)$.
- The student identifies domain intervals where f increases or decreases.
- The student sketches a graph of $y = f(x)$.
- The student provides domain intervals upon which the function is continuous.

1f. analyze basic trigonometric functions. Students are fluent with the properties and behavior of sine, cosine, and tangent. Students identify zeros and their corresponding intercepts as well as asymptotes. Students provide exact values when appropriate and otherwise provide estimations. *

Illustrations:

- Given $s(x) = \sin(2x)$, the student determines all zeros of s, graphs $y = s(x)$, and, determines intervals where $s(x)$ increases and decreases.
- Given $c(x) = \cos\left(\frac{x}{2}\right)$, the student determines all zeros of c. Graphs $y = c(x)$, and determines intervals where $c(x)$ increases and decreases.
- The student becomes fluent with conversions using traditional equivalency families. (e.g., $(\sin(t))^2 + (\cos(t))^2 = 1$; $(\tan(t))^2 + 1 = (\sec(t))^2$; double angle rules; half-angle rules)
- The student provides domain intervals upon which a function is continuous.
- Given the model $h(t) = 6\tan\left(\frac{t}{12}\right)$ which gives height in feet of a model rocket off the ground t seconds after liftoff, the student determines a reasonable applied domain for the model.

2. Algebraic Properties: Successful Technical Mathematics students apply basic algebraic properties of function types to produce equivalent forms appropriate for the current question or investigation.

The successful Technical Mathematics student can:

2a. apply the distributive property effectively. Students decide when to factor or expand expressions to support a particular goal or enhance communication. Students identify common factors and construct equivalent products when seeking zeros. *

Illustrations:

- Student factors $p(x) = 6x^2 - x - 1$ to identify zeros.
- The student presents a strategy for solving a given equation based on its algebraic structure.
- The student differentiates the meaning of parentheses within algebraic expressions involving functions.
- After identifying forms, the student selects and applies methods from previous courses.
- To graph $f(x) = \frac{(x^2-4)}{(x-2)}$, the student simplifies to $f(x) = x + 2$ and graphs $y = x + 2$ with a hole at $(2,4)$.
- Given the graph of what appears to be a parabola, the student can use knowledge of the vertex and intercepts to create a formula for the function in the form $f(x) = a(x-h)^2 + k$ or $f(x) = ax^2 + bx + c$ or $f(x) = a(x - r_1)(x - r_2)$.

2b. communicate fluently with the language of Algebra. Students view expressions from the details of individual components to the patterns in which those components are suspended. Students attend to the broad algebraic structure before addressing details. *

Illustrations:

- The student translates a function from a verbal description to an algebraic description to determine its domain.
- The student recognizes the equation $e^{2f} - e^f = 5$ as quadratic in form and uses the quadratic formula to solve for e^f.
- The student recognizes the equation $(\cos(\theta))^2 - \sin(\theta) = 5$ as quadratic in form and uses the quadratic formula or factoring to solve for $\sin(\theta)$, then determines values of θ.
- The student applies characteristics of functions to reason that an equation does not have a solution.
- The student rephrases an equation as information about a function and then identifies solutions based on function characteristics.
- The student explains the difference between the quantities $f(x+h)$ and $f(x) + h$.
- The student explains the difference between $f(x) + f(x)$ and $f(x) + f(y)$.
- The student combines $\frac{3}{(p-2)} + \frac{5}{(p+3)}$ into a single faction.

2c. manipulate exponents. Students condense, combine, and expand exponential expressions. *

Illustrations:

- To solve $\sqrt{\cos(t)} = \sqrt{\sin(t)}$, the student solves $\cos(t) = \sin(t)$ and knows this procedure may result in extraneous solutions.

- Student selects appropriate exponent rules and apply them.
- Student provides counterexamples to incorrect exponent rules.
- The student solves $3\sqrt{2x-9} + 3x\frac{1}{2}(2x-9)^{-\frac{1}{2}}2 = 0$

2d. manipulate logarithms. Students condense, combine, and expand logarithmic expressions. Students convert exponential-logarithmic compositions. *

Illustrations:

- Student rewrites $g(x) = 2 - \ln(\sqrt{x}) - 2\ln(x^2)$ as a single logarithm for analysis.
- Student solves $\log_2(y^2) = 3$
- Student selects and applies appropriate logarithmic rules.

2e. manipulate radicals. Students condense, combine, and expand radical expressions. Students swap radical expressions for exponential expressions and vice-versa. *

Illustrations:

- Student converts expressions from radical to exponential notation.
- Student simplifies $h(t) = \sqrt{t+1} + \frac{t}{2}\frac{1}{\sqrt{t+1}}$
- Student simplifies $B(x) = \frac{\sqrt{x+1} - \frac{x}{2}\frac{1}{\sqrt{x+1}}}{(\sqrt{x+1})^2}$
- Student selects and applies appropriate radical rules.
- Student simplifies $k(m) = \sqrt{e^{2m} + 2 + e^{-2m}}$

2f. utilize trigonometric identities. Students can use basic trigonometric identities to construct equivalent expressions. *

Illustrations:

- Student solves $\sin(\theta) = -\cos(\theta)$
- Student solves $\sqrt{1-\cos^2(\theta)} = \sin(2\theta)$
- Student solves $2\sin^2(t) + 7\sin(t) - 4 = 0$
- Student explains the difference between $sin(2x)$ and 2 $sin(x)$.
- Student follows simplification of difference quotient involving Trigonomotric functions.

3. Rate of Change: Successful Technical Mathematics students communicate effectively about situational rates as well as rates-of-change encoded within functions.

The successful Technical Mathematics student can:

3a. calculate rates of change over intervals. Students calculate rate-of-change and interpret. *

Illustrations:

- Given the formula for a rational function f, the student simplifies the difference quotient for f.
- Student calculates the rate-of-change of $f(x) = x\, e^{x-3}$ over the interval $[1,4]$.
- From graph, student identifies places in the domain where the rate-of-change is greatest or least.
- From graph, student identifies places in the domain where the rate-of-change is 0.
- From a linear rate-of-change graph, student calculates accumulated change via the area under curve.

3b. linearize a function around a point. Students plot tangent lines to graphs of functions and create linear functions that approximate the original function. *

Illustrations:

- Given the graph of $y = f(x)$, containing the point $(3,5)$, student draws the tangent line at $(3,5)$ and creates an equation for the tangent line.
- Student Interprets the tangent line to the graph of f as the graph of the linear function $L(x)$, approximate the value of $f(3.2)$.

3c. represent rate-of-change as a function. Students sketch approximate graphs for rate-of-change of a function. Use these to describe function behavior. *

Illustrations:

- Given the graph of $y = f(x)$, student sketches a graph of the rate-of-change over $[x, x+\varepsilon]$.
- Student connect maximum and minimum values of $f(x)$ with zeros of the rate-of-change.
- Student deduces that the composition of linear function is linear, and the rate-of-change is the product of the rates-of-change.

4. Composition: Successful Technical Mathematics students are comfortable with the operations of functions, especially composition.

The successful Technical Mathematics student can:

4a. compose functions. Students compose functions algebraically. Students can identify component functions from a given composition. Students can establish domains and ranges of compositions. *

Illustrations:

- Student analyzes compositions involving known functions:

$$\log(\sin(t)),\, e^{\cos(x)}, \sqrt{\cos(\theta)}\,, \sin\left(\frac{1}{\varphi}\right), e^{-t}\sin(2u)$$

- Student creates $f(x)$, such that $f(3x+1) = 6x - 5$
- Student verbally describes the relationship between the graph of $y = f(x)$, and each of $y = f(|x|)$ and $y = |f(x)|$.
- Given an algebraic description for f and a graph for g, the student determines values for the sum, difference, product, quotient, and composition of f and g.
- Given the graph of f and g, the student determines the domain of $\frac{f}{g}$ and $f \circ g$.
- Given formulas for $f(\theta)$ and $(f \circ g)(\theta)$, the student creates a formula for $g(\theta)$.
- The student finds functions f, g, and h so that $F(t) = \sqrt{\frac{3t-4}{t+1}} = \left(f \circ \left(\frac{g}{h}\right)\right)(t)$ or $F(t) = \sqrt{\frac{3\cos(t)-4}{\cos(t)+1}} = \left(f \circ \left(\frac{g}{h}\right)\right)(t)$.

4b. employ composition as an operation. Students view the identity function as the identity element in composition. Students can construct inverse functions and produce the identity function via composition. *

Illustrations:

- Student determines the inverse of $f(x) = \frac{6-x}{2x+5}$ and implied domain.
- Student explains why the reciprocal of a function is not the inverse function.
- Given $g(x) = \frac{x-2}{(x-4)(x+5)}$, student determines the different inverse formulas depending on domain.
- Student sketches graph of $y = g^{-1}(x)$, given graph of $y = g(x)$.
- Student graphs $arctan(t)$ and $arcsin(t)$
- Student applies $arctan(t)$ and $arcsin(t)$ when solving equations.

4c. express linear composition graphically. Students interpret linear compositions in terms of graphical transformations. Students encapsulate graphical transformations algebraically. *

Illustrations:

- Given the graph of a function f, the student can graph $y = 3f(1-x) + 2$.
- Given the graphs of two functions, the student can determine if they appear to be related by a sequence of transformations.
- Student determines formula for the composition of two piecewise linear functions.
- Student creates the conversion between Fahrenheit and Celsius.
- Student creates a piecewise linear function which models the cost of a mobile phone data plan.
- Student recognizes linear trends in data and create a piecewise linear function to model behavior.

5. Analytical Geometry: Successful Technical Mathematics students are comfortable with various descriptions and interpretations of curves, especially circles and ellipses.

The successful Technical Mathematics student can:

5a. connect Cartesian equations and graphs of ellipses. Students can plot ellipses from equations. Students can create equations from plotted ellipses. *

Illustrations:

- Student draws the curve described by $\frac{x^2}{2^2} + \frac{y^2}{3^2} = 1$.
- Student rewrites $x^2 - 4x + 4y^2 = 0$ in standard form in order to graph.

5b. parameterize ellipses. Students parameterize ellipses with sine and cosine.*

Illustrations:

- Student converts between a Cartesian description $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and a parameterization $(a\cos(\theta), b\sin(\theta))$.

5c. parameterize lines. Students analyze multiple parameterizations for the same line. *

Illustrations:

- Student converts the parameterization $\begin{matrix} x(t) = 3t + 1 \\ y(t) = 2t - 3 \end{matrix}$ to the Cartesian equation $2x - 3y = 11$.
- Student identifies points on a line corresponding to values of the parameter.