TMM005 – Calculus I
(Updated August 23, 2018)

Traditionally, a Calculus I course has been described by the content it presents to students. However, recommendations from projects including *Common Vision*[1] as well as reports such as the *Committee on the Undergraduate Program in Mathematics*[2] tell us that content makes only half a successful course. A successful Calculus I course must also give equal consideration to student engagement. This revision of the TMM005 guidelines follows the spirit of these recommendations. Although the content of Calculus I remains essentially as it was, this content is rephrased through an active learning lens.

The guidelines listed below do not alter the material content of Calculus I in any significant manner. Calculus I still includes a numerical, graphical, and algebraic investigation of limits; functional interpretations of limits; continuity; derivative definition, rules, and theorems; graphical interpretations of the derivative as well as rates of change between variables; higher-order derivatives; curve sketching; function analysis; optimization; and an introduction to integral calculus including antiderivatives, areas of planar regions, substitution, and the Fundamental Theorem of Calculus. All of this content is again included and described in the guidelines below. The goal of this revision is to phrase these in terms of student engagement and student outcomes.

While the material content of Calculus I remains steady, we hope our Calculus I courses continue to evolve. As the *Common Vision* report cites "the status quo is unacceptable." By intentionally rephrasing the guidelines, we hope to spark ideas for the other half of the course – student engagement. To that end, there are an overwhelming number of sample tasks phrased from a student perspective. These are not included as requirements. There is no implication of coverage. There is no suggested weighting. They are not directed at the content half of the course. They reference the engagement half of the course. They are included as a first step towards elevating our thinking of active learning, student engagement, and student outcomes in Calculus I. Placement of sample tasks is not a stipulation nor a restriction. The sample tasks are simply encouragement to faculty to elevate the importance of student engagement and suggest a level of learning. Ultimately, TMM005 courses should expose our students to the same material and also collectively engage our students in this material as much as possible. The nouns of calculus have long been established. We now need to promote the verbs.

Successful Calculus students demonstrate a deep understanding of functions whether they are described verbally, numerically, graphically, or algebraically (both explicitly and implicitly). Students proficiently work in detail with the following families of functions: linear, quadratic, higher-order polynomial, rational, exponential, logarithmic, radical, trigonometric, piecewise-defined functions, and combinations, compositions, and inverses of these. Students are adept with the tools of differentiation and integration and their application toward situational goals. Students articulate the relationships underlying rate-of-growth and accumulation. Finally, successful Calculus students offer observations, suggestions, and conclusions to an investigative discussion as well as respond to remarks by others.

In a Calculus I (TMM005) course, students should:

- develop effective thinking and communication skills;
- operate at a high level of detail;
- state problems carefully, articulate assumptions, understand the importance of precise definition, and reason logically to conclusions;
- identify and model essential features of a complex situation, modify models as necessary for tractability, and draw useful conclusions;
- deduce general principles from particular instances;
- use and compare analytical, visual, and numerical perspectives in exploring mathematics;
- assess the correctness of solutions, create and explore examples, carry out mathematical experiments, and devise and test conjectures;
- recognize and make mathematically rigorous arguments;
- read mathematics with understanding;
- communicate mathematical ideas clearly and coherently both verbally and in writing to audiences of varying mathematical sophistication;
- approach mathematical problems with curiosity and creativity, persist in the face of difficulties, and work creatively and self-sufficiently with mathematics;
- learn to link applications and theory;
- learn to use technological tools; and
- develop mathematical independence and experience open-ended inquiry.

– Adapted from the MAA/CUPM 2015 Curriculum Guide

To qualify for TMM005 (Calculus I), a course must achieve all of the following essential learning outcomes listed in this document (marked with an asterisk). These make up the bulk of a Calculus I course. Courses that contain only the essential learning outcomes are acceptable from the TMM005 review and approval standpoint. It is up to individual institutions to determine further adaptation of additional course learning outcomes of their choice to support their students' needs. In addition, individual institutions will determine their own level of student engagement and manner of implementation. These guidelines simply seek to foster thinking in this direction.

1. **Limits and Continuity:** Successful Calculus students demonstrate a deep understanding of the concepts of limit and continuity whether described verbally, numerically, graphically, or algebraically (both explicitly and implicitly). At the foundational level, students examine images and preimages of open intervals while comparing properties as the intervals contract. Illustrating and explaining this mapping is at the heart of understanding limits. At the procedural level, students identify common hurdles within the limiting process and select appropriate algebraic options. At the functional level, students categorize limit properties, recognize functions involved in limiting situations, and strategically invoke the broader principles of continuity when establishing consequences.

The successful Calculus student can:

1a. analyze limits graphically and numerically. The student visually describes the limiting process with graphs of functions. The student produces/reads tables of function values which illustrate limit properties.*

Sample Tasks:

- The student concludes nested intervals are converging to a number or not converging to a number.
- The student visually explains how nested domain intervals correspond to nested range interval and guess limiting values.
- Given a graph of the piecewise defined function
$$G(t) = \begin{cases} 3t+5 & t < 2 \\ 2t+1 & 2 \le t \end{cases}$$
the student explains why the image of every open interval around 2 in the domain contains function values greater than 5.
- Given the graph of a function, a range number, r, and an open interval around r on the vertical axis, the student draws an open interval on the horizontal axis that maps into the range interval.
- The student describes numbers that are getting closer to d, but greater than (or less than) d, and where they are positioned on the horizontal axis. The student maps these to the vertical axis.
- The student reads/writes limit notation ($\lim_{x \to 5} f(x) = 8$) and connects the notational pieces to the graph of a function.
- The student produces graphs for which $H(5) = 7$, yet $\lim_{k \to 5} H(k) \neq 7$, $\lim_{k \to 5^+} H(k) \neq 7$, and/or $\lim_{k \to 5^-} H(k) \neq 7$.
- The student produces graphs for which $H(5) \neq 7$, yet $\lim_{k \to 5} H(k) = 7$.
- The student uses technology to extend the examination to functions defined implicitly via equations. For example, student graphs a curve defined via an equation, chooses a section of curve that defines a function, hypothesizes limiting values based on graph, and then uses this information to help with an algebraic examination.
- The student uses a graphing utility[1] to compare limiting behavior of different families of functions. For example, compare limiting behavior of polynomial and exponential functions. For instance, $\lim_{x \to \infty} \frac{x^{1000000000000000000000000000}}{e^x}$.
- The student discusses "near" in a limiting context.
- The student illustrates limiting process with tables of function values.

- The student illustrates failure of limiting process with tables of function values.
- The student illustrates the difference between two-sided and one-sided limits with tables of function values.
- The student uses a computer algebra system[4] to produce tables of values with accuracies unavailable via calculators or by hand. For example, $\lim_{x\to\infty} \frac{x^{10000}}{e^x} = 0$. Provide a lower limit on x such that $\frac{x^{10000}}{e^x} < 0.5$.
- The student uses a computer algebra system[4] to create domain intervals that map into cited range intervals. For example, if $\lim_{t\to 8.7} (t-6.5)^5(t+8)^2(t-9) = L$, then find d such that the domain interval $(8.7-d, 8.7+d)$ maps into the range interval $(L-0.01, L+0.01)$.

1b. find limits algebraically. The student organizes a well-formed presentation of the details involved in the limiting process via formulas.*

Sample Tasks:

- The student explicitly applies limit rules using correct notation throughout.
- The student explains a plan of algebraic manipulation for evaluating a limit, before executing the plan.
- The student describes algebraic hurdles that might arise for particular types of formulas.
- The student categorizes types of formulas with types of techniques.
- The student explains the value of techniques (e.g., how does a conjugate help?).
- The student analyzes limits of piecewise-defined functions.
- The student explains how indeterminate forms can have a limiting value.
- The student algebraically evaluates limits of the form $\lim_{h\to 0} \frac{f(a+h)-f(h)}{h}$.
- The student states which limit rules they are using as they proceed through a computation.
- The student identifies the assumptions of limit rules and can provide counterexamples that justify the need for those assumptions.

1c. recognize and explain limits at infinity. The student furthers his/her understanding of infinity and how to logically work with unboundedness.*

Sample Tasks:

- The student explains the "existence" of M such that $\lim_{t\to\infty} g(t) = 7$ describes behavior of $g(t)$ for the interval (M, ∞).
- The student classifies functions in order of dominance for limits at infinity.
- The student carries out algebraic limits at infinity.
- The student compares asymptotic behavior to simpler elementary functions.

1d. communicate fluently about the concept of continuity.*

Sample Tasks:

- The student explicitly applies the limit definition of continuity.
- The student constructs logical arguments for discontinuities.
- The student creates functions where different aspects of the definition fail.
- The student develops an argument about the continuity at 0 for the following functions and verbally explains the differences:

$$s_0(\theta) = \begin{cases} \sin\left(\frac{1}{\theta}\right) & \theta \neq 0 \\ 0 & \theta = 0 \end{cases}$$

$$s_1(\theta) = \begin{cases} \theta \sin\left(\frac{1}{\theta}\right) & \theta \neq 0 \\ 0 & \theta = 0 \end{cases}$$

- The student defends the assertion that $R_1(x) = \frac{1}{x}$ is continuous on its implied domain while $R_2(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ is not.

- The student discusses possible images of connected and disconnected pieces of the domain.
- The student classifies different types of discontinuities.
- The student hypothesizes on the continuity of the composition of functions.
- The student draws graphs of functions which demonstrate the need for each requirement in the Intermediate Value Theorem.
- The student uses a graphing utility[3] to produce discontinuous functions whose graphs look continuous.
- The student points out limitations to technology. For example,

$$s(\theta) = 5 + 0.00000001 \sin(\theta)$$

2. Differentiation: Successful Calculus students demonstrate an extensive understanding of the concept of differentiation from the details of specific procedures to the logical reasoning of abstracting relationships. Students are comfortable with the algebraic details presented by the definition of the derivative and differentiation rules. Students manipulate algebraic representations to reveal and explain properties and characteristics of derivatives, explicitly and implicitly. Students include properties and characteristics of the derivative into their analysis of situational models. Students articulate the necessity of derivative conditions in abstract reasoning.

The successful Calculus student can:

2a. apply the definition of the derivative to differentiate a function at a number and extend to an interval, choose appropriate differentiation rules and apply them, and parse formulas for application of the chain rule.*

Sample Tasks:

- The student evaluates $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ for the number 'a' in the domain.
- The student interacts with $D(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ as a formula for a function.
- The student summarizes possible problems with the derivative definition and corresponding graphical characteristics.
- The student relates the differences between continuity and differentiation.
- The student paraphrases the proof of rules of differentiation.
- The student devises formulas for functions whose derivative matches a given formula.
- The student applies rules in succession obtaining derivatives of higher order.
- The student uses a graphing utility[3] to differentiate between the derivative at a number vs. over an interval. Graph $f'(x)$ and $g(x)=\frac{f(x+h)-f(x)}{h}$ over an interval for various values of h. Discuss the size of h to bring the approximation to within a tolerance over the interval. Discuss intervals where the approximation is better and worse.
- The student uses a graphing utility[3] to illustrate the difficulty of differentiation at domain numbers associated with corners in graphs or discontinuities in piecewise-defined functions. For example, graph the derivative of the absolute value function.

2b. operate the derivative as a tool. The derivative measures rates of change, and the student is able to utilize this tool within the framework of a functional model, including connecting the slope of a tangent line with the value of the derivative.*

Sample Tasks:

- The student builds an equation of the tangent line using derivative value.
- The student speculates about the change in function values relative to the change in domain values based on values of the derivative.
- The student explains meaning of rate of change in context.
- The student relates average rate of change to instantaneous rate of change.
- The student formulates a description for a linear approximating function.
- The student approximates function values.
- The student describes increasing and decreasing over an interval.

- The student compares increasing/decreasing over a domain interval to increasing/decreasing at a number in the domain.
- The student expresses the importance of open intervals and interval investigation in general.
- The student partitions the domain into pieces where the function is increasing, decreasing, or constant.
- The student phrases rate of change in terms of independent and dependent variables.
- The student compares changes in functions within a parameterized scenario.
- The student builds a function model for measured characteristics within a situation.
- The student translates situational restrictions or requirements into aspects of the modelling function.
- The student transfers function analysis back to applications and arrives at contextual conclusions and consequences.

2c. work with implicitly defined functions. The student begins to expand his/her idea of function beyond a representation where the dependent variable is isolated on one side of an equation.*

Sample Tasks:

- The student partitions a curve into pieces that defined a functional relation.
- The student discovers characteristics of a curve that dictate elements of partition.
- The student cites domain and range restrictions for implicit function definition.
- The student uses an equation description of a curve to evaluate function values.
- The student uses an equation description of a curve to evaluate derivative values.
- The student uses a parameterization of a curve to evaluate function values.
- The student uses a parameterization of a curve to evaluate derivative values.
- The student can switch orientation by switching the roles of the independent and dependent variables.
- The student constructs linear approximations.
- The student deduces the rule for the derivative of an inverse function from $\frac{dy}{dx} \cdot \frac{dx}{dy} = 1$.
- The student describes situations by relating rates of change and analyzes such situations.
- The student explains relationships between the growth rates of dependent and independent measurements.
- The student discusses relationships between the growth rates of parameterized quantities.
- The student uses a graphing utility[3] to visually suggest domain and range restrictions in creating implicitly defined functions. For example, graph the Siluroid Folium, $(x^2 + y^2)^2 - 4\,x\,n\,(x^2 - y^2) = 0$ - identify sections of the curve that define y as a function of x and approximate values of $\frac{dy}{dx}$ for values of x. Identify points where a domain interval cannot be created where y is defined as a function of x. Switch roles: view y as the independent variable and x the dependent variable.
- The student uses a graphing utility[3] to investigate the relation between curves and tangent lines.
- The student uses a graphing utility[3] to illustrate difficulties related to tangent lines at particular points on a curve and relates the curve characteristics of such points back to the derivative. What is the equation of the tangent line at $(0, 0)$ on $y = |x|$?

3. Graphing and Optimization: Successful Calculus students fully analyze situations described by functions. Students develop their own strategies and tactics and explain how their plan will coalesce into applicable information. Students summarize their plans, including statements of the situational goal; recognize the types of functions involved; appropriately apply derivatives, limits, and function properties; offer reasoning for their choices and decisions when information was sought; and purposefully arrive at conclusions.

The successful Calculus student can:

3a. identify critical numbers and extrema values. The student separates function characteristics from features of the graph.*

Sample Tasks:

- The student states an explicit definition of a global maximum/minimum.
- The student states an explicit definition of a local maximum/minimum.
- The student applies a definition to justify maximum/minimum values of a piecewise-defined function.

  Justify that $H(y) = \begin{cases} 3y+2 & y \le 4 \\ -y+3 & y > 4 \end{cases}$ has a maximum at 4, but $R(y) = \begin{cases} 3y+2 & y < 4 \\ -y+3 & y \ge 4 \end{cases}$ does not.

- The student identifies endpoints of intervals and places of discontinuity as places where extrema occur or fail.
- The student explains how the derivative is used to locate critical numbers.
- The student produces a graph of a function with given characteristics.
- The student explains the ideas behind the first and second derivative tests.
- The student applies the first and second derivative test.
- The student uses a graphing utility[3] to suggest places to look for extrema.
- The student uses a graphing utility[3] to illustrate the derivative tests. For example, graph the derivative of $k(t)$ and identify/classify extrema candidates for $k(t)$. For instance, graph the derivative of $R(y) = \begin{cases} 3y+2 & y < 4 \\ 3y-3 & y \ge 4 \end{cases}$

3b. sketch curves/graphs of functions using derivatives and limits. The student acquires a level of competency with visual representations. This is crucial to using technology meaningfully.*

Sample Tasks:

- The student organizes his/her own plan to analyze a function including:
  - intervals where a function is increasing/decreasing,
  - concavity,
  - global/local maximums/minimums,
  - asymptotes,
  - discontinuities, and
  - intercepts.
- The student presents a coherent report on his/her plan for analysis.
- The student draws appropriate conclusions and visually represents them.
- The student convinces other students of the validity of his/her plan and subsequent graph.
- The student uses technology to suggest points of focus for analysis.
- The student uses technology for comparison and contrast to his/her reasoning.
- The student identifies the limits of technology and explains how to decode the correct information from the display. For example, fluctuations too small for the display of the graphing calculator. Or, fluctuations that coincide with the pixel dimensions and thus hide the fluctuations.

3c. optimize quantities in applied problems. The student develops some fluency with the application of calculus tools to physical situations modeled by functions.*

Sample Tasks:

- The student organizes his/her own plan to optimize a function.
- The student presents a coherent report of his/her conclusions.
- The student judges conclusions.

4. Integration: Successful Calculus students can reverse the differentiation process. Working symbolically, students recognize the algebraic result of the chain rule, parse expressions into pieces based on their compositional position, and formulate a reasonable antiderivative. Students can quickly apply the chain rule to their suggested antiderivative, identify differences, and effectively alter their antiderivative. As tools, students use definite integrals to describe the accumulation of changes and the antiderivative to measure accumulation. Students communicate this relationship between rates of change and accumulation via the Fundamental Theorem of Calculus.

The successful Calculus student can:

4a. create antiderivatives. The student is fluent with the differentiation – antidifferentiation relationship and uses the FTC and integration by substitution.*

Sample Tasks:

- Given a derivative formula, the student maps out which part was produced by the chain rule.
- Given a derivative, the student creates a possible original function.
- The student can articulate his/her strategies for identifying chain rule outcomes within integrand.
- The student generalizes characteristics to families of antiderivatives.
- The student creates his/her own auxiliary functions with respect to the chain rule and, via substitution, rewrites the integral to better adhere to chain rule structure.
- The student converts integration limits according to substitution function.
- The student recovers format of original function after successful integration via substitution.
- The student investigates "+C" on separate connected components of the domain.
  For example, the antiderivative of $f(x) = \frac{1}{x}$ is $F(x) = \begin{cases} \ln(-x) + C & x < 0 \\ \ln(x) + D & x > 0 \end{cases}$

4b. measure area of bounded planar regions. Given a planar region whose boundary curves are described by equations, the student describes the situation in terms of functions and accompanying integration setup.*

Sample Tasks:

- The student symbolizes area measurement with a definite integral.
- The student approximates area measurement via Riemann sums.
- The student evaluates limit of Riemann sums for value of definite integral and area.
- The student measures area with a definite integral.
- The student uses a computer algebra system[4] to calculate Riemann sums and hypothesize. For example, identify a value for n such that Riemann sum is within a given tolerance of area value. Compare Riemann sums for different choices of evaluation number inside subintervals.

4c. understand the Fundamental Theorem of Calculus (FTC). The FTC connects the measurements of rates of change and accumulation. Understanding this relationship is intrinsic to an understanding of calculus.*

Sample Tasks:

- The student paraphrases the FTC and its purpose.
- The student calculates area measurement via definite integral and FTC.
- The student operates $F(x) = \int_a^x f(t)\, dt$ as a function of x.
- The student hypothesizes about the derivative of $F(x) = \int_a^x f(t)\, dt$.
- The student uses a computer algebra system[4] to investigate and hypothesize about the FTC.
- The student uses technology to explain the importance of the intervals in the FTC.
- The student uses technology to explain the importance of continuity in the FTC.
- The student uses technology to explain the meaning of differentiability in the FTC.
- The student uses technology to hypothesize on these connections. For example, graph the derivative of $F(x) = \int_2^x 3t^2 - 2t + 5\, dt$ and $f(x) = 3x^2 - 2x + 5$. Or, graph $\frac{F(x+h)-F(x)}{h}$ and $f(x) = 3x^2 - 2x + 5$ for various values of h.

[1] https://www.maa.org/sites/default/files/pdf/CommonVisionFinal.pdf

[2] https://www.maa.org/sites/default/files/CUPM%20Guide.pdf

[3] Graphing utility includes online apps, downloadable apps, computer software, graphing calculator, along with other possibilities.

[4] Computer algebra system includes online apps, downloadable apps, computer software, graphing calculator along with other possibilities.