TMM006 – Calculus II
(Updated August 23, 2018)

Traditionally, a Calculus II course has been described by the content it presents to students. However, recommendations from projects including *Common Vision*[1] as well as reports such as the *Committee on the Undergraduate Program in Mathematics*[2] tell us that content makes only half a successful course. A successful Calculus II course must also give equal consideration to student engagement. This revision of the TMM006 guidelines follows the spirit of these recommendations. Although the content of Calculus II remains essentially as it was, this content is rephrased through an active learning lens.

The guidelines listed below do not alter the material content of Calculus II in any significant manner. Calculus II still includes using antiderivatives to evaluate definite integrals; a variety of applications to model physical, biological, or economic situations; geometric applications emphasizing Riemann sums; integration techniques such as substitution, by parts, trigonometric substitution, and partial fractions; evaluating indeterminate forms, including L'Hôpital's Rule; improper integrals; sequence and series convergence, including comparison, ratio, root, integral, and alternating tests; Taylor polynomials; power series representation, differentiation, and integration; and curve analysis whether described parametrically or in polar form along with area measurement of regions bounded by such curves. All of this content is again included and described in the guidelines below. The goal of this revision is to phrase these in terms of student engagement and student outcomes.

While the material content of Calculus II remains steady, we hope our Calculus II courses continue to evolve. As the *Common Vision* report cites "the status quo is unacceptable." By intentionally rephrasing the guidelines, we hope to spark ideas for the other half of the course – student engagement. To that end, there are an overwhelming number of sample tasks phrased from a student perspective. These are not included as requirements. There is no implication of coverage. There is no suggested weighting. They are not directed at the content half of the course. They reference the engagement half of the course. They are included as a first step towards elevating our thinking of active learning, student engagement, and student outcomes in Calculus II. Placement of sample tasks is not a stipulation nor a restriction. The sample tasks are simply encouragement to faculty to elevate the importance of student engagement and suggest a level of learning. Ultimately, TMM006 courses should expose our students to the same material and also collectively engage our students in this material as much as possible. The nouns of calculus have long been established. We now need to promote the verbs.

Successful Calculus students demonstrate a deep understanding of functions whether they are described verbally, numerically, graphically, or algebraically (both explicitly and implicitly). Students proficiently work in detail with the following families of functions: linear, quadratic, higher-order polynomial, rational, exponential, logarithmic, radical, trigonometric, piecewise-defined functions, and combinations, compositions, and inverses of these. Students are adept with the tools of differentiation and integration and their application toward situational goals. Students articulate the relationships underlying rate-of-growth and accumulation. Finally, successful Calculus students offer observations, suggestions, and conclusions to an investigative discussion as well as respond to remarks by others.

In a Calculus II (TMM006) course, students should:

- develop effective thinking and communication skills;
- operate at a high level of detail;
- state problems carefully, articulate assumptions, understand the importance of precise definition, and reason logically to conclusions;
- identify and model essential features of a complex situation, modify models as necessary for tractability, and draw useful conclusions;
- deduce general principles from particular instances;
- use and compare analytical, visual, and numerical perspectives in exploring mathematics;
- assess the correctness of solutions, create and explore examples, carry out mathematical experiments, and devise and test conjectures;
- recognize and make mathematically rigorous arguments;
- read mathematics with understanding;
- communicate mathematical ideas clearly and coherently both verbally and in writing to audiences of varying mathematical sophistication;
- approach mathematical problems with curiosity and creativity, persist in the face of difficulties, and work creatively and self-sufficiently with mathematics;
- learn to link applications and theory;
- learn to use technological tools; and
- develop mathematical independence and experience open-ended inquiry.

– Adapted from the MAA/CUPM 2015 Curriculum Guide

To qualify for TMM006 (Calculus II), a course must achieve all of the following essential learning outcomes listed in this document (marked with an asterisk). These make up the bulk of a Calculus II course. Courses that contain only the essential learning outcomes are acceptable from the TMM006 review and approval standpoint. It is up to individual institutions to determine further adaptation of additional course learning outcomes of their choice to support their students' needs. In addition, individual institutions will determine their own level of student engagement and manner of implementation. These guidelines simply seek to foster thinking in this direction.

1. **Applications of Definite Integral:** Successful Calculus students model aspects of situations with functions. When the measurements under investigation are best described in terms of accumulation, then calculus students distinguish the varying characteristic, identify relationships, and build integral functions that measure the desired situational aspect. Calculus students describe and explain their integration models. Calculus students participate in discovery activities where they criticize and defend ideas.

The successful Calculus student can:

1a. model geometric measurements such as area, volume of solids of revolution, arc length, area of surfaces of revolution, and centroids with integration tools, including setting up an approximating Riemann sum and representing its limit as a definite integral.*

Sample Tasks:

- The student interprets and describes accumulation with Riemann sums.
- The student constructs definite integrals as limits of Riemann sums.
- The student models areas of 2-dimensional regions bounded by curves described via Cartesian equations, polar equations, or within parameterized coordinate systems.
- The student models volumes of 3-dimensional regions bounded by surfaces described via Cartesian equations, polar equations, or within parameterized coordinate systems.
- The student calculates area and volumes by applying antiderivative rules.
- The student explains connections between integrand expressions and given situations involving arc length or area of surface.
- The student designs integrals for length or surface area measurements.
- The student visualizes and predicts location of centroids.
- The student measures centroid coordinates via definite integrals.
- The student criticizes choice of coordinates or integration variable.

1b. model measurements within situations related to STEM fields.*

Sample Tasks:

- The student describes a measurement of work as a Riemann sum.
- The student describes a measurement of work as definite integral.
- The student adapts Riemann sums to accumulation situations.
- The student formulates models with respect to situations involving fluid forces in terms of Riemann sums.
- The student verbalizes Riemann sum structure within physical, biological, and economic situations.

1c. approximate accumulation measurements. [This guideline is not required, but many institutions include this in various ways as a foundational competency within their course.]

Sample Tasks:

- The student formulates definite integrals as a limit of Riemann sums.
- The student illustrates how definite integrals are approximated with finite sums.
- The student discusses error size.
- The student demonstrates the application of the Trapezoidal Rule.
- The student leads a group of students in the application of Simpson's Rule.
- The student participates in a group comparing Runge-Kutta algorithms using a computer.

2. Integration Techniques: Successful Calculus students demonstrate an extensive understanding of the concept of integration from the details of specific procedures to the logical reasoning of abstracting relationships. Students are comfortable with the algebraic manipulation used to rewrite integrands and integration limits. Students can articulate their choices and decisions when applying integration techniques.

The successful Calculus student can:

2a. use antiderivatives to evaluate definite integrals and employ a variety of integration techniques, including substitution, integration by parts, trigonometric substitution, and partial fraction decomposition.*

Sample Tasks:

- The student relates the benefits of chosen substitutions.
- The student criticizes substitution choices and predicts their ineffectiveness.
- The student identifies algebraic hurdles within integrands.
- The student speculates on types of techniques appropriate to algebraic hurdles.
- The student forecasts a list of steps to follow and expected outcomes.
- The student chooses appropriate integration techniques and applies them logically.
- The student classifies types of integrands according to techniques.
- The student illustrates application of integration by parts.
- The student demonstrates application of trigonometric substitution.
- The student explains application of partial fraction decomposition.

3. Improper Limits: Successful Calculus students are comfortable dealing with infinity. Students have an intuition about how functions compare in their end-behavior or near singularities. Students can precisely phrase situations involving infinity.

The successful Calculus student can:

3a. identify indeterminate forms within limits. The student reasons on his/her own that an indeterminate form is present. The student chooses an appropriate action. The student executes his/her plan and explains the whole process.*

Sample Tasks:

- The student analyzes function for possible singularities.
- The student classifies types of indeterminate forms.
- The student recognizes types of indeterminate forms.
- The student hypothesizes on limiting behavior.
- The student chooses and explains appropriate algebraic plan.
- The student justifies the application of L'Hôpital's Rule.
- The student correctly concludes if L'Hôpital's Rule is applicable.
- The student reasons and predicts if a limit exists or not.
- The student writes complete analysis with correct notation.

3b. identify and evaluate improper integrals, including integrals over infinite intervals and integrals in which the integrand becomes infinite in the interval of integration. The student deduces on his/her own that a given integral is improper. The student rephrases the integral precisely using limits and develops and executes a plan for calculation. The student presents the whole process.*

Sample Tasks:

- The student analyzes the integrand for possible singularities.
- The student recognizes if the interval of integration includes singularities.
- The student recognizes if the integrand end-behavior affects the integral.
- The student rewrites improper integrals with limits.
- The student reasons and predicts if integral will converge or diverge.
- The student writes complete analysis with correct notation.
- The student compares like integrands for similar behavior.

4. Sequences and Series: Successful Calculus students can analyze objects defined as limits. Calculus students are developing a working relationship with the infinite. They are experiencing how properties of limiting objects can differ from objects cited in the defining process. Students' language and communication is evolving, especially in its precision. Sequence and series present a platform for the analysis of limiting numbers and functions.

The successful Calculus student can:

4a. critically analyze and discuss numerically, graphically, algebraically, verbally, and in other relevant ways a sequence or series of numbers. The student understands the difference between convergence and the limiting value and can determine convergence by using appropriate tests.*

Sample Tasks:

- The student explains what it means for a sequence of numbers to converge.
- The student presents a sequence as a function from the whole numbers (or integer subset).
- The student creates a graph of a sequence.
- The student calculates sequential values at given indices.
- The student distinguishes sequences from series.
- The student illustrates a series as a sequence of partial sums.
- The student hypotheses on convergence.
- The student speculates on choice of convergence test and summarizes expectations.
- The student conducts convergence test, explicitly satisfying conditions, and quoting conclusion.
- The student expands convergence concepts and procedures to include parameters and operate series as a function.
- The student separates questions of convergence from questions of limiting values.
- The student recognizes geometric series and can determine their limiting value.

4b. critically analyze and discuss numerically, graphically, algebraically, verbally, and in other relevant ways a sequence or series of functions, including Taylor and power series and associated error terms. The student understands the difference between convergence and the limiting function and can determine convergence by using appropriate tests. The student can apply the reasoning and techniques of Calculus with series representations *

Sample Tasks:

- The student explains what it means for a sequence of functions to converge to a function.
- The student presents a sequence of functions as a function from the whole numbers (or integer subset).
- The student creates a sequence of graphs for the component functions.
- The student evaluates functional values at given indices.
- The student distinguishes sequences from series.
- The student illustrates a series as a sequence of partial sums.
- The student debates properties of limiting function.

- The student illustrates properties of limiting function not present in any component/summand function.
- The student estimates domain and range of limiting function.
- The student hypothesizes on interval of convergence.
- The student creates n^{th} Taylor polynomial.
- The student calculates error terms and expresses their meaning.
- The student compares Taylor series graphically.
- The student conducts convergence test and states domain of limiting function.
- The student moves the center of expansion and identifies the radius of convergence interval.
- The student creates series expansions around centers using existing series.
- The student explains a Taylor series that does not converge to the original function.
- The student differentiates limiting function.
- The student integrates limiting function.

5. Curves: Successful Calculus students understand the difference between curves and functions. Students heading to multi-variable calculus as well as STEM fields will encounter functions using ordered tuples as domain elements. Students must be able to change their perspective on curves depending on the situation. Sometimes students should interpret a tuple component as function information and use these for function analysis. Sometimes it is not intended that the components possess an independent-dependent variable relationship. The curve is a collection of points with geometric properties. Further, these points may have been collected via auxiliary functions and may be used as domain information for multi-variable functions.

The successful Calculus student can:

5a. think parametrically. The student interacts with graphs with different perspectives depending on the situation.*

Sample Tasks:

- The student draws curve described parametrically.
- The student understands the difference between a curve and a parameterization.
- The student traces locations based on parameter behavior.
- The student calculates parameter values associated to curve properties.
- The student draws a curve with xy-axes or with other coordinate axes.
- The student converts coordinates from one system to another.
- The student theorizes on modifications to reverse parameterization.
- The student hypothesizes on modifications to shift parameterization.
- The student explains how to parameterize a line in a direction with an initial point.
- The student proposes parameterization for a given ellipse.
- The student can analyze curves expressed with polar coordinates.
- The student can convert coordinates between systems.

5b. measure area of bounded planar regions. Given a planar region whose boundary curves are described by equations, the student describes the situation in terms of functions and accompanying integration setup.*

Sample Tasks:

- The student draws boundary and shades region in given coordinate system.
- The student identifies limits of integration with respect to coordinate system.
- The student symbolizes area measurement with a definite integral.
- The student calculates area with a definite integral.

[1] https://www.maa.org/sites/default/files/pdf/CommonVisionFinal.pdf

[2] https://www.maa.org/sites/default/files/CUPM%20Guide.pdf